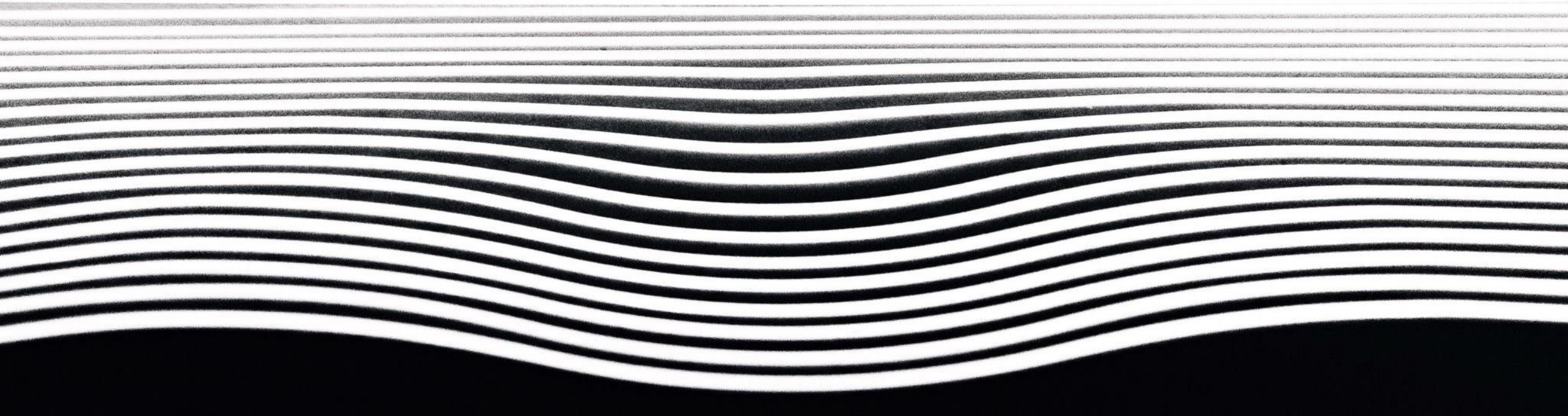


Numerical methods for conservation laws

1: Introduction



Lecturer: Martin Licht, martin.licht@epfl.ch

Lecture: Wednesday, 13:15-15:00

Tutor: Fernando Henriquez

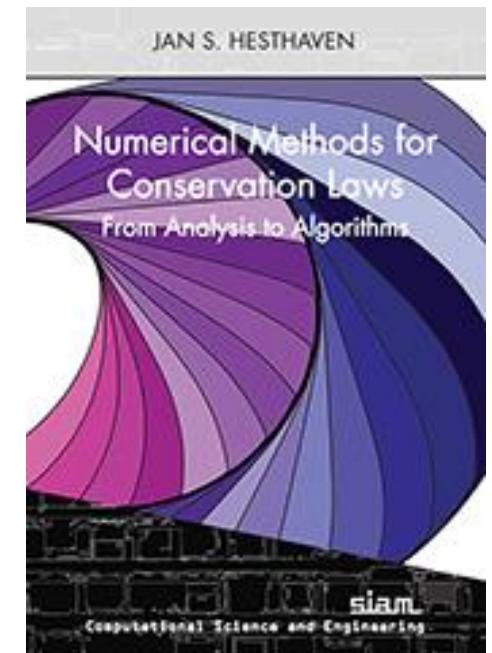
Exercises: Friday, 15:15-17:00

Textbook: *Jan S. Hesthaven, Numerical Methods for Conservation Laws*

Exercises: Published on Moodle no later than Thursday after the lecture

Slides will be on Moodle before the lecture (except this one).

Lecture videos are published (more or less) frequently on Youtube.



For the purpose of this class, a conservation law is a partial differential equation (PDE) of the form

$$\partial_t u + \operatorname{div} f(u, x, t) = 0$$

time derivative

divergence in x

flux

This is a first-order evolution problem.

Generally speaking, this class studies:

- Existence & Uniqueness (weak solutions, entry conditions)
- Qualitative properties (shock waves, physics background)
- Numerical methods (Finite Diff., Finite Volume, DG methods, ...)

General remarks

- Evolution problem, i.e., time-dependent. Hence we need initial data at $t = 0$
- We know initial data and the flux. We want to know $u = u(x, t)$
- More generally, the flux may be dependent on the time:

$$f(u) = f(u, x, t)$$

- The unknown u can be a scalar (scalar conservation law) or may have multiple components (system of conservation laws)
- In practice, we look at a spatial domain $\Omega \subseteq \mathbb{R}^d$ and some time interval $[0, T]$.

Example: scalar conservation law in 1D

We consider a quantity over a spatial interval

Let $u(x, t)$ be the density of a substance at point x and time t

The total amount over the subinterval $[x_1, x_2]$ is

$$M(t) = \int_{x_1}^{x_2} u(x, t) \, dx$$

Let $a(x, t)$ denote the velocity of the substance at (x, t) , then physics gives us

$$\partial_t \int_{x_1}^{x_2} u(x, t) \, dx = a(x_1, t)u(x_1, t) - a(x_2, t)u(x_2, t) = - \int_{x_1}^{x_2} \partial_x (a(x, t)u(x, t)) \, dx$$

Example: scalar conservation law in 1D

Consequently,

$$\int_{x_1}^{x_2} \partial_t u(x, t) dx = - \int_{x_1}^{x_2} \partial_x (a(x, t) u(x, t)) dx$$

This identity is valid at any time t and over any subinterval $[x_1, x_2]$.

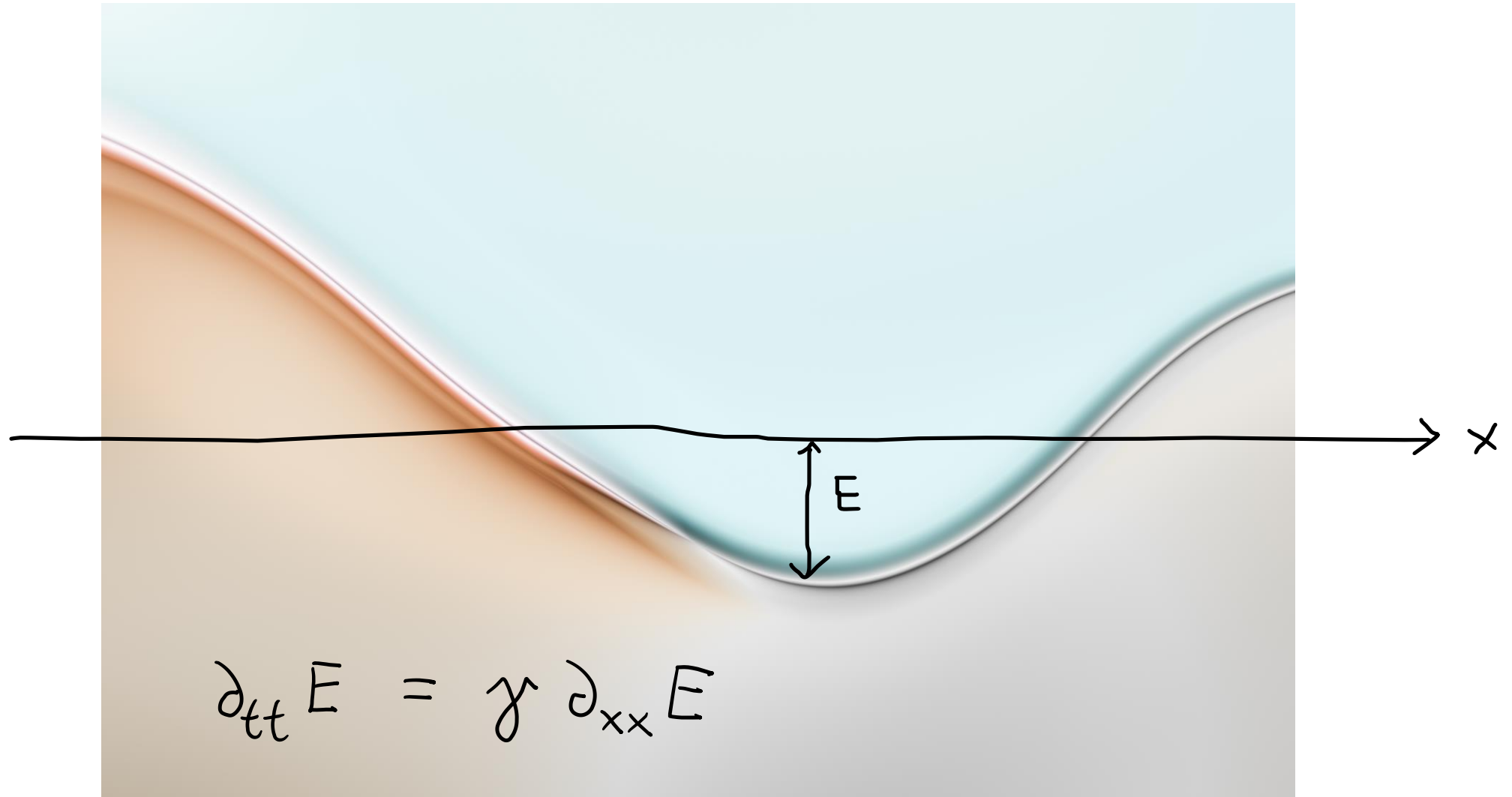
Therefore we deduce the conservation law

$$\partial_t u(x, t) = -\partial_x (a(x, t) u(x, t))$$

This is a conservation law. What is the flux?

$$f(u, x, t) = a(x, t) u$$

Example: wave equation in 1D



Example: wave equation in 1D

Recall the wave equation in 1D, without external forces:

$$\partial_{tt}E - \gamma \partial_{xx}E = 0$$

We introduce auxiliary variables for the first derivatives of E, say,

$$v = \partial_t E, \quad s = \partial_x E$$

These satisfy the equations

$$\begin{aligned}\partial_t s - \partial_x v &= 0 \\ \partial_t v - \gamma \partial_x s &= 0\end{aligned}$$

$$\partial_t \begin{pmatrix} s \\ v \end{pmatrix} + \partial_x \begin{pmatrix} 0 & -1 \\ -\gamma & 0 \end{pmatrix} \begin{pmatrix} s \\ v \end{pmatrix} = 0$$


$$\vec{u} = \begin{pmatrix} s \\ v \end{pmatrix}$$

$$f(\vec{u}, x, t) = \begin{pmatrix} 0 & -1 \\ -\gamma & 0 \end{pmatrix} \vec{u}$$

Why “Conservation Law”?

Let $\omega \subseteq R^d$ be a small domain and let $u(x, t)$ describe the density of some quantity distributed in space. Then we consider

$$\int_{\omega} u(x, t) dx$$

How does this amount change over time?

$$\partial_t \int_{\omega} u(x, t) dx = ?$$

Often, there is a flux field $f(u, x, t)$ that describes the flux of the quantity at any given position and time.

Why “Conservation Law”?

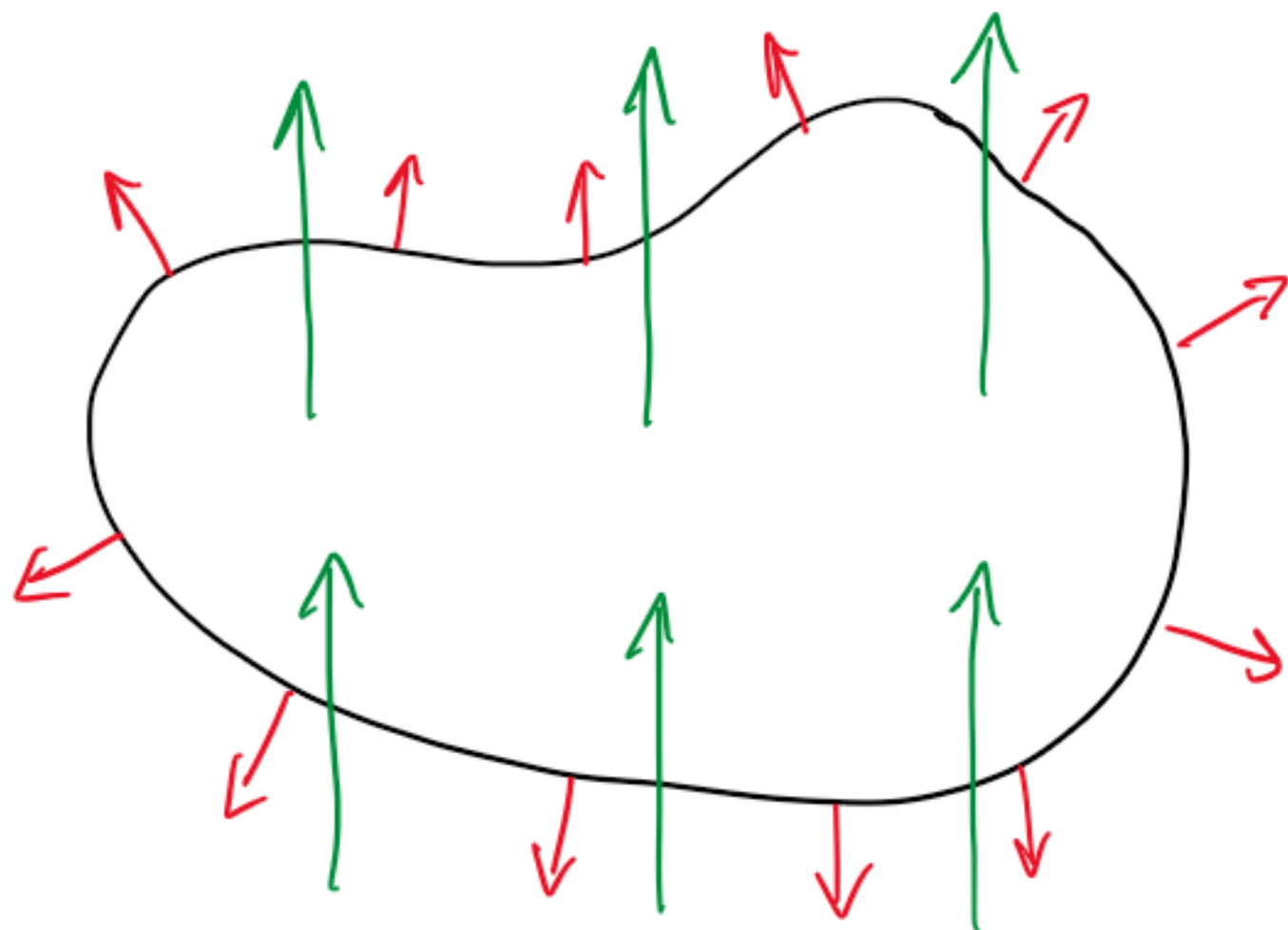
Physical intuition: the negative divergence of the flux describes how much of the quantity is created or lost.

$$\partial_t \int_{\omega} u(x, t) \, dx = \int_{\omega} \partial_t u(x, t) \, dx = - \oint_{\omega} f(u, s, t) \cdot \vec{n} \, ds = - \int_{\omega} \operatorname{div} f(u, x, t) \, dx$$

This relates to the balance equation

$$\partial_t \int_{\omega} u(x, t) \, dx = - \oint_{\omega} f(u, s, t) \cdot \vec{n} \, ds$$

In practice, such balance equations lead to the conservation law PDE.



Flux in more detail

If u is a scalar function,

$$u : R^d \times R \rightarrow R$$

$\times \quad t$

Then the flux must be

$$f : R \times R^d \times R \rightarrow R^d$$

$u \quad \times \quad t$

The flux f takes values in a vector field, and we take the divergence.

If u has multiple (say, m) components,

$$u : R^d \times R \rightarrow R^m$$

$\times \quad t$

Then the flux must be

$$f : R^m \times R^d \times R \rightarrow R^{m \times d}$$

$u \quad \times \quad t$

The flux takes values in matrices of size $m \times d$. We take the divergence of this matrix rowwise.

