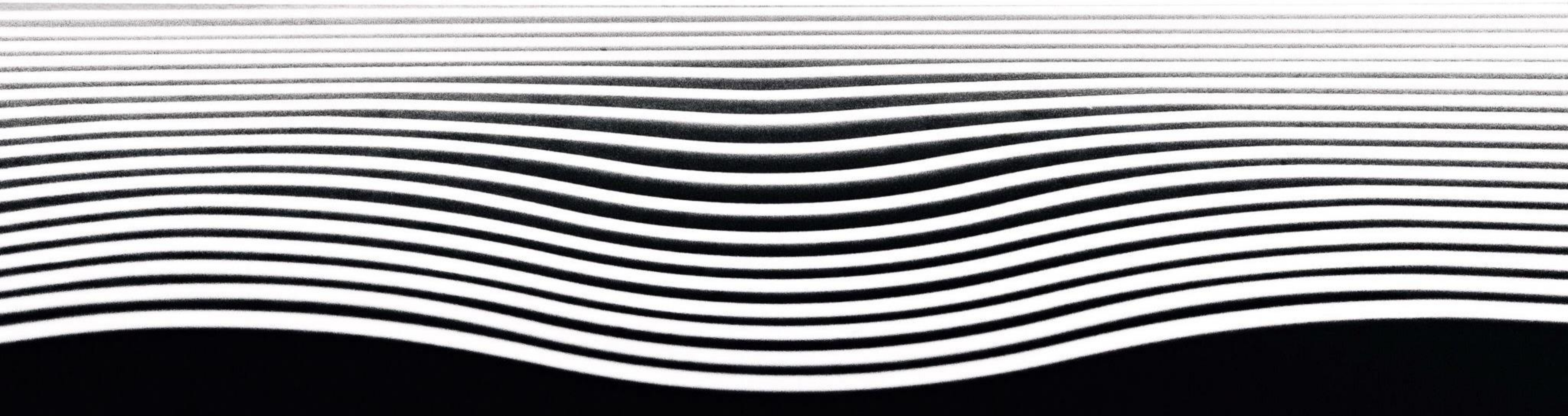
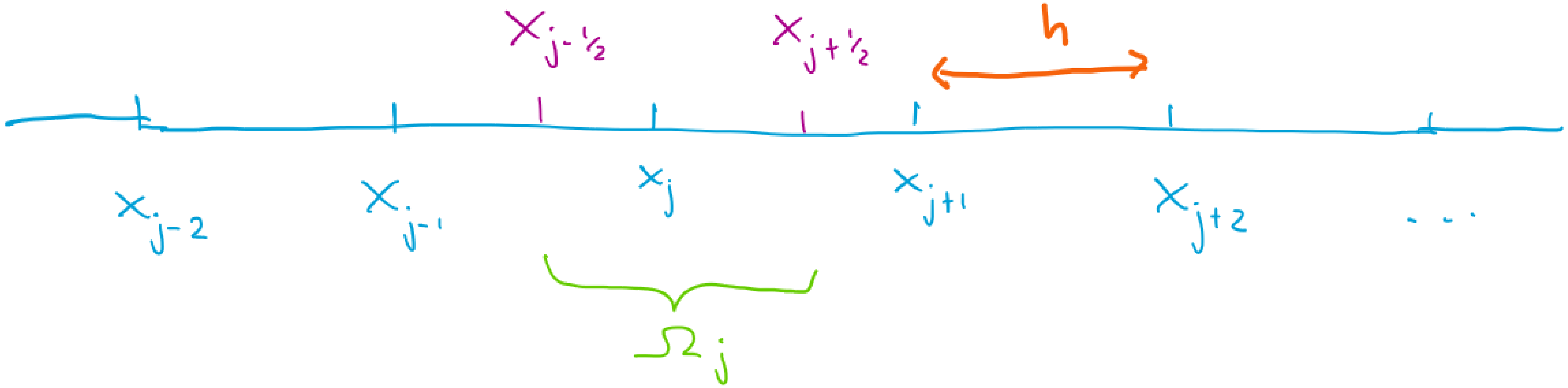
Numerical methods for conservation laws
15: Essentially non-oscillatory schemes

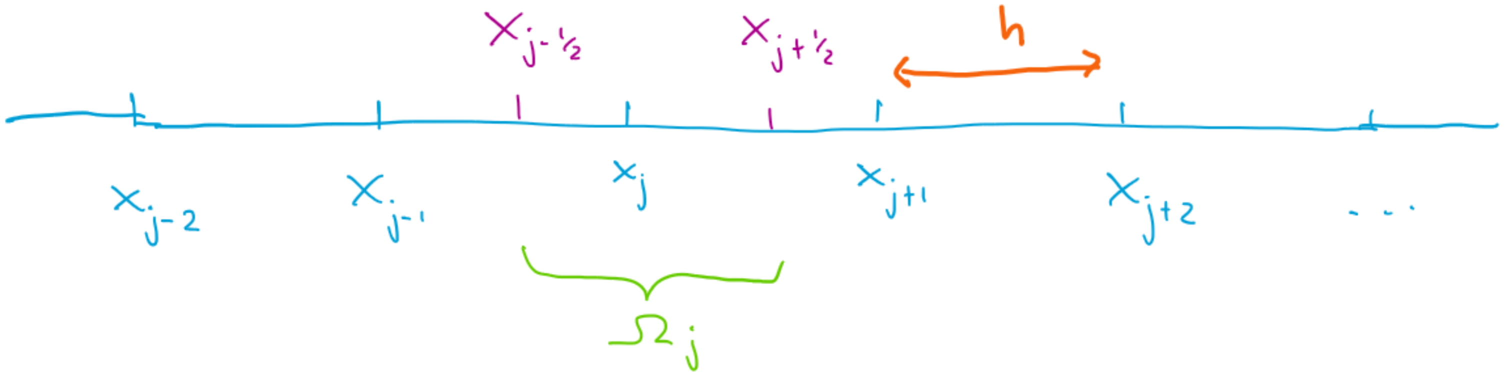




Suppose we have a function u, sufficiently smooth, and only know its cell averages:

$$\bar{U}_j^n = \frac{1}{h}\int_{\Omega_j} u(x)dx$$

Can we reconstruct the value of u at the cell interfaces (approximately)?



Given cell averages

$$\bar{U}^n_{j-p}, \cdots \bar{U}^n_{j+q},$$

we seek a polynomial π of degree $m-1$, where $m = p+q+1$ is the number of averages, satisfying the m constraints

$$\bar{U}^n_l = \frac{1}{h}\int_{\Omega_l} \pi(x)dx\,, \qquad l = j-p,\ldots,j+q$$

Ultimately, we hope to achieve an approximation quality of

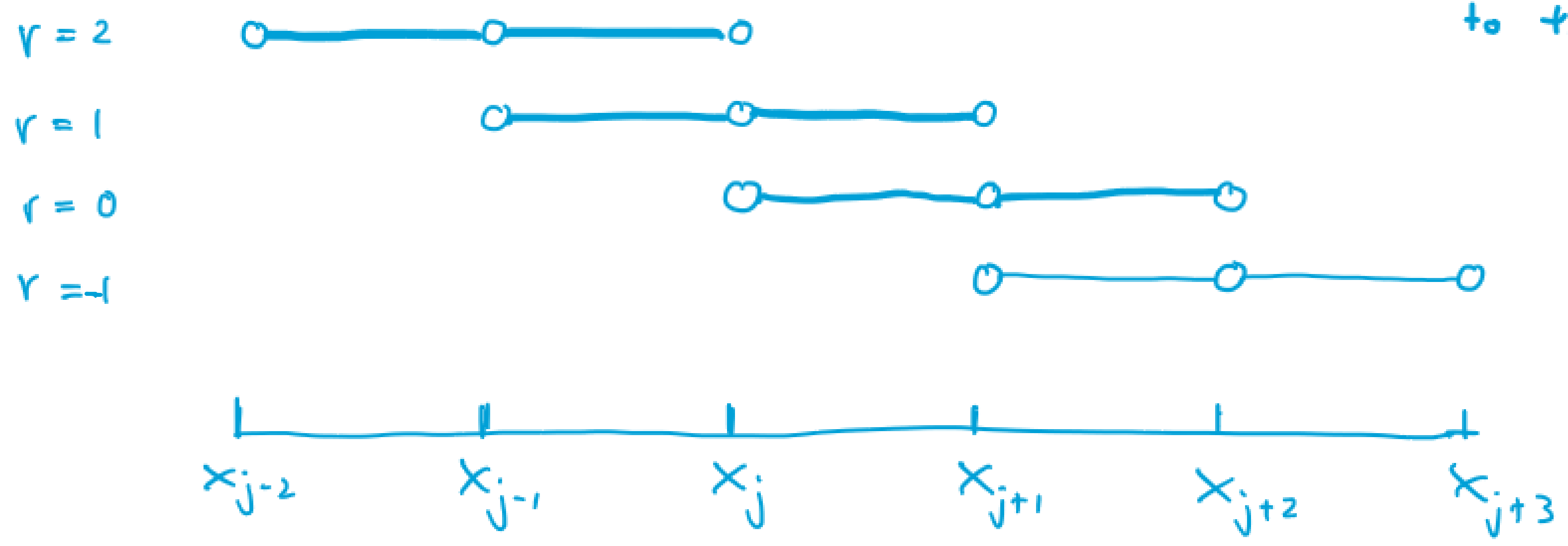
$$u\left(x_{j+\frac{1}{2}}\right) = \pi\left(x_{j+\frac{1}{2}}\right) + O(h^m)$$

We consider numerous interpolation stencils that take m averages into account and which differ by a shift parameter

$$j-r, \qquad j-r+1, \qquad \cdots, \qquad j-r+m-1$$

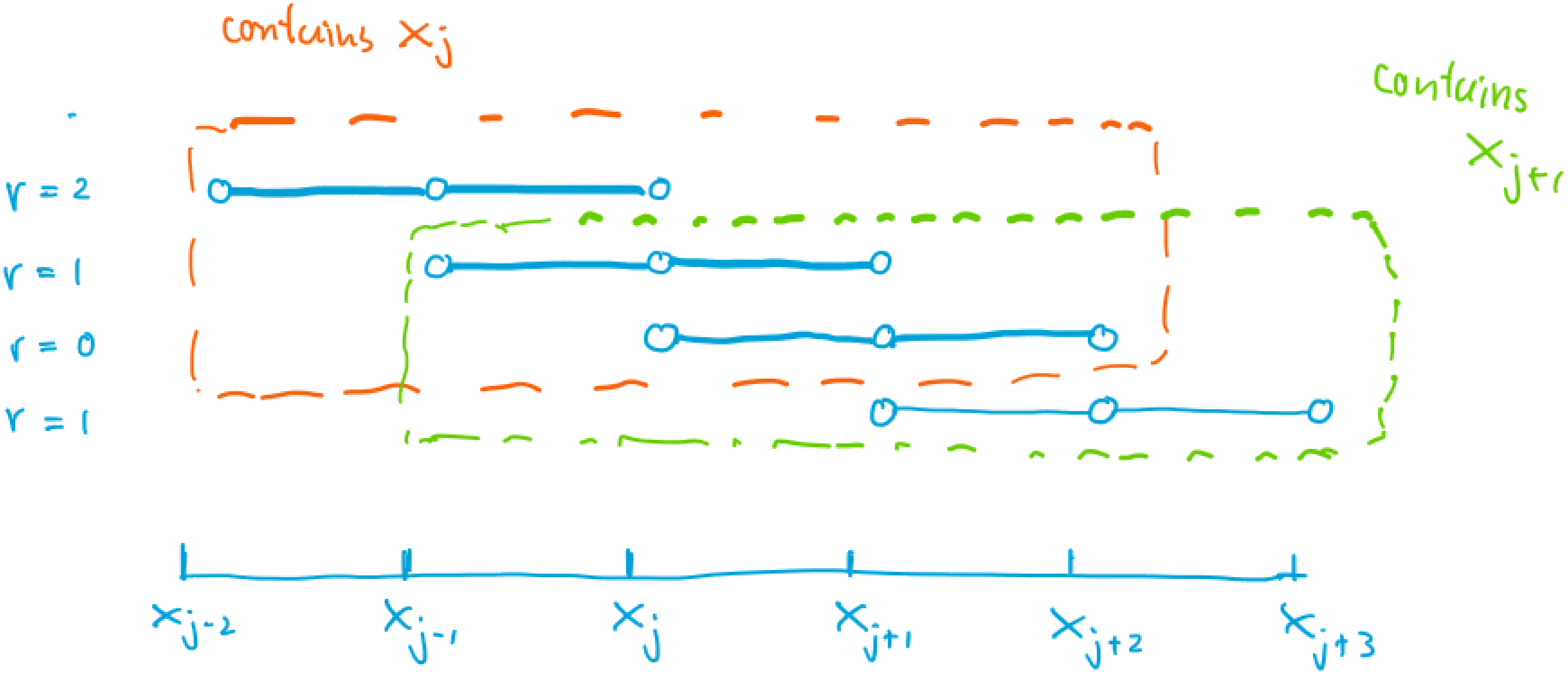
$\boxed{\text{Example}}$ $m = 3$

r = shift parameter to the left



Note: In order to contain x_j or x_{j+1} in the stencil, the shift must satisfy

$$-1 \le r \le m - 1.$$



We have two families of stencils, each of size m

Interlude: polynomial approximation

Theorem:

Let $f \in C^{m+1}[a, b]$ and let Π be the degree m polynomial satisfying

$$f(x_i) = \Pi(x_i)$$

at interpolation points $x_0, \ldots, x_m \in [a, b]$. Then for each $x \in [a, b]$ there exists $\xi_x \in [a, b]$ such that

$$f(x) - \Pi(x) = \frac{1}{(m+1)!} f^{(m+1)}(\xi_x) \prod_{i=0}^{m} (x - x_i)$$

Consequence: the interpolation is bounded in terms of f's derivative of order $m + 1$ over the interval $[a, b]$.

We prove the approximation property. Consider any antiderivative $V(x)$ of $u(x,t)$, that is, $V'(x) = u(x)$.

We fix a shift $-1 \leq r \leq m-1$.

We let $\Pi(x)$ be the polynomial of degree m that interpolates $V(x)$ at the cell interfaces:

$$\Pi\left(x_{j-r+i-\frac{1}{2}}\right) = \mathrm{V}\left(x_{j-r+i-\frac{1}{2}}\right), \qquad i = 0, \dots, m$$

We inspect

$$\pi(x) = \Pi'(x)$$

We calculate:

$$\frac{1}{h}\int_{x_{\ell-1/2}}^{x_{\ell+1/2}} \pi(\xi)\, d\xi = \frac{1}{h}\int_{x_{\ell-1/2}}^{x_{\ell+1/2}} \Pi'(\xi)\, d\xi$$

$$= \frac{1}{h}\left(\Pi(x_{\ell+1/2}) - \Pi(x_{\ell-1/2}) \right)$$

$$= \frac{1}{h}\left(V(x_{\ell+1/2}) - V(x_{\ell-1/2}) \right)$$

$$= \frac{1}{h}\int_{x_{\ell-1/2}}^{x_{\ell+1/2}} u(\xi)\, d\xi = \overline{u}_\ell$$

Therefore, π satisfies the desired constraints.

Over the entire interval $[x_{j-r-1/2}, x_{j-r-m-1/2}]$:

$$\Pi = V + O(h^{m+1})$$

Over the entire interval $[x_{j-r-1/2}, x_{j-r-m-1/2}]$:

$$\pi = u + O(h^{m})$$

So the polynomial π defined by the averages does satisfy the approximation property at the interface.

We can compute the polynomial π of degree $m-1$ using m constraints. **But is there an explicit solution formula?**

We recall the explicit formula for the polynomial interpolant, using Lagrange polynomials:

$$\Pi\left(x_{j-r+i-\frac{1}{2}}\right) = \mathrm{V}\left(x_{j-r+i-\frac{1}{2}}\right), \qquad\qquad i = 0, \dots, m$$

means that

$$\Pi(x) = \sum_{i=0}^{m} \mathrm{V}\left(x_{j-r+i-\frac{1}{2}}\right) \cdot l_i^{\left(j-r-\frac{1}{2}\right)}(x)$$

Here, the Lagrange polynomials are defined by

$$l_i^{\left(j-r-\frac{1}{2}\right)}(x) = \prod_{\substack{0 \le t \le m \\ t \ne i}} \frac{x - x_{j-r+t-\frac{1}{2}}}{x_{j-r+i-\frac{1}{2}} - x_{j-r+t-\frac{1}{2}}}$$

Important properties include the Lagrange property and the partition of unity property:

$$l_i^{\left(j-r-\frac{1}{2}\right)}\left(x_{j-r+t-\frac{1}{2}}\right) = \delta_{it}, \qquad 1 = \sum_{i=0}^{m} l_i^{\left(j-r-\frac{1}{2}\right)}(x)$$

Using that representation of Π and the partition of unity formula:

$$\Pi(x) - \mathrm{V}\left(x_{j-r-\frac{1}{2}}\right) = \sum_{i=0}^{m} \left(\mathrm{V}\left(x_{j-r+i-\frac{1}{2}}\right) - \mathrm{V}\left(x_{j-r-\frac{1}{2}}\right) \right) \cdot l_i^{\left(j-r-\frac{1}{2}\right)}(x)$$

Using the fundamental theorem of calculus once more, we get

$$\Pi(x) - \mathrm{V}\left(x_{j-r-\frac{1}{2}}\right) = \sum_{i=0}^{m} \left(h \sum\nolimits_{q=0}^{i-1} \bar{U}_{q+j-r} \right) \cdot l_i^{\left(j-r-\frac{1}{2}\right)}(x) = h \sum_{i=0}^{m} \sum\nolimits_{q=0}^{i-1} \bar{U}_{q+j-r} \cdot l_i^{\left(j-r-\frac{1}{2}\right)}(x)$$

We take the derivative:

$$\pi(x) = h \sum_{i=0}^{m} \sum\nolimits_{q=0}^{i-1} \bar{U}_{q+j-r} \cdot \frac{d}{dx} l_i^{\left(j-r-\frac{1}{2}\right)}(x)$$

Recall that we can rearrange nested sums

$$\sum_{i=0}^{m}\sum_{q=0}^{i-1} a_q b_i = \sum_{q=0}^{m-1}\left(a_q \sum_{i=q+1}^{m} b_i\right)$$

The reconstructed polynomial

$$\pi(x) = \sum_{q=0}^{m-1} \bar{U}_{q+j-r} \underbrace{\sum_{i=q+1}^{m} h \frac{d}{dx} l_i^{\left(j-r-\frac{1}{2}\right)}(x)}$$

can be evaluate at the interface points:

$$\pi\left(x_{j+1/2}\right) = \sum_{q=0}^{m-1} \bar{U}_{q+j-r} \underbrace{\sum_{i=q+1}^{m} h \frac{d}{dx} l_i^{\left(j-r-\frac{1}{2}\right)}\left(x_{j+1/2}\right)}_{\boldsymbol{C_{q,r}^m}}$$

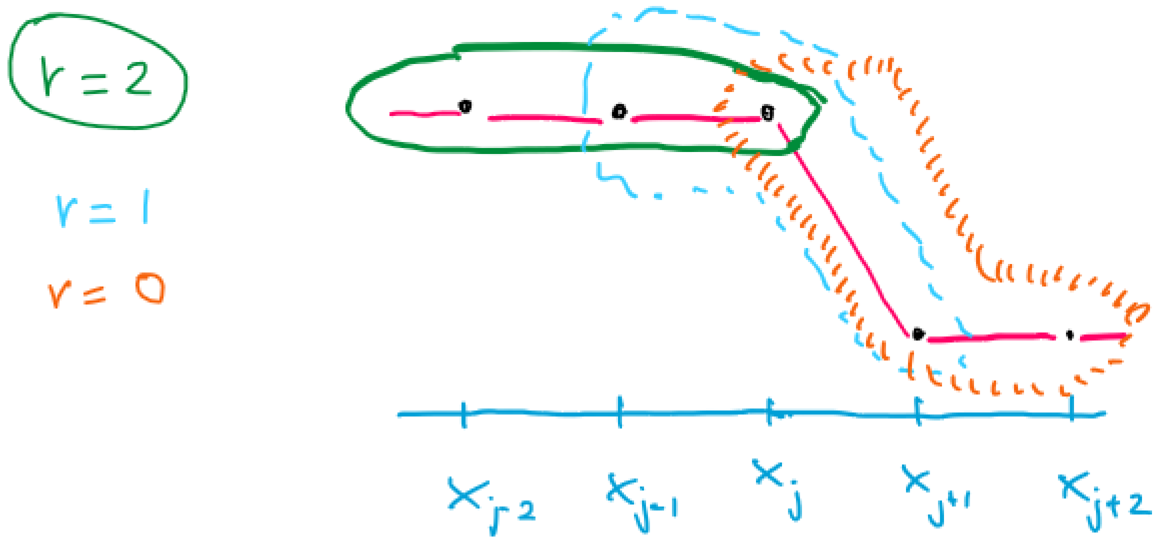
The terms $\boldsymbol{C_{q,r}^m}$ only depend on the parameters h, $\boldsymbol{m}$, $\boldsymbol{l}$, $\boldsymbol{r}$, but not on $\boldsymbol{j}$.

Thus we have a concise formula for the interpolated interface values:

$$u\left(x_{j+1/2}\right) = \sum_{q=0}^{m-1} \bar{U}_{q+j-r} \boldsymbol{C_{q,r}^m} + \boldsymbol{O}(\boldsymbol{h^m})$$

How are we going to use this?

- If r is fixed, then we get a linear scheme.
- Instead, we choose r adaptively
- We try to choose the shift to obtain a stable and accurate scheme. As a rule of thumb, we try to pick the shift such that jumps are avoided.



Fine, how do we choose the stencils?

We measure the smoothness over each stencil in terms of divided differences.

Suppose we have nodal points $x_0, x_1, \ldots, x_m$ with distance h.

- $f[x_i] = f(x_i)$
- $f[x_i, x_{i+1}] = \dfrac{f(x_{i+1}) - f(x_i)}{x_{i+1} - x_i} = \dfrac{f[x_{i+1}] - f[x_i]}{x_{i+1} - x_i} \approx f'(x_i)$
- $f[x_i, x_{i+1}, x_{i+2}] = \dfrac{f[x_{i+1}, x_{i+2}] - f[x_i, x_{i+1}]}{x_{i+2} - x_i}$

$$= \frac{1}{2} \frac{f[x_{i+2}] - 2 f[x_{i+1}] + f[x_i]}{h^2} \approx \frac{1}{2} f''(x_i)$$

$\vdots$

- $f[x_i, \ldots, x_{i+l}] = \dfrac{f[x_{i+1}, \ldots, x_{i+l}] - f[x_i, \ldots, x_{i+l-1}]}{x_{i+l} - x_i} \approx \dfrac{1}{m!} f^{(m)}(x_i)$

Divided differences can be used to measure the "smoothness" of a discrete function.

$$f[x_0, x_1] \begin{cases} \approx f'(x_0) & \text{if } f \text{ is diff'able} \\ O(h^{-1}) & \text{if } f \text{ has a jump} \end{cases}$$

More generally

$$f[x_0, \dots, x_m] \begin{cases} \frac{1}{m!} f^{(m)}(x_0) & \text{if } f \text{ is diff'able} \\ O(h^{-m}) & \text{if } f \text{ has a jump} \end{cases}$$

Application to stencil selection

The magnitude of divided differences gives an idea of the smoothness of a stencil. How do we implement this?

We begin with

$$S_0 = \{x_j\}$$

Iteratively, given a stencil

$$S_l = \{x_{j-p}, \cdots, x_{j+q}\}$$

we consider

$$S_0^- = \{x_{j-p-1}, x_{j-p}, \cdots, x_{j+q}\}$$
$$S_0^+ = \{x_{j-p}, \cdots, x_{j+q}, x_{j+q+1}\}$$

We either let $S_{l+1} = S_0^-$ or $S_{l+1} = S_0^+$, depending on which of the two stencils has the smallest divided difference.

Remark: alternatively, we may consider all possible stencils and pick the one with the best "smoothness" measure. However, that requires computing all relevant divided difference quotients.

Application to stencil selection

This constructs the stencils containing x_j. We thus compute the interface values contributed by each cell:

$$U_{j+1/2}^{-} = \sum_{i=0}^{m-1} C_{r,i}^{m} \cdot \overline{U}_{j-r+i}$$

possibly different stencil shifts

$$U_{j-1/2}^{+} = \sum_{i=0}^{m-1} C_{s-1,i}^{m} \cdot \overline{U}_{j-s-1+i}$$

We use those approximate interface values in our conservative scheme:

$$\overline{U}_j^{n+1} = \overline{U}_j^{n} - \frac{k}{h}\left(F\left(U_{j+1/2}^{+}, U_{j+1/2}^{-}\right) - F\left(U_{j-1/2}^{+}, U_{j-1/2}^{-}\right) \right)$$

Summary and remarks:

- We have introduced essentially non-oscillatory (ENO) schemes
- Looks at first like a linear high-order scheme to reconstruct the value of u at the interface (the numerical flux might be nonlinear); but we adaptive shift the stencil, hence very nonlinear behavior.
- Generally speaking, ENO schemes work well in practice but our theoretical understanding is limited.