

PROJECT 1. THE FINITE DIFFERENCE METHOD FOR THE SHALLOW WATER EQUATION

Consider the one-dimensional *shallow water* equation

$$\begin{bmatrix} h \\ m \end{bmatrix}_t + \begin{bmatrix} m \\ \frac{m^2}{h} + \frac{1}{2}gh^2 \end{bmatrix}_x = \mathbf{S}(x, t), \quad (1)$$

where $h = h(x, t)$ is the *depth* or the height of the water, $m = m(x, t)$ is a quantity – usually called the *discharge* in shallow water theory – which measures the flow rate of the fluid past a point, g is the acceleration due to gravity, and $\mathbf{S}$ is some source term. Note that (1) has the form $\mathbf{q}_t + \mathbf{f}(\mathbf{q})_x = \mathbf{S}(x, t)$ with $\mathbf{q} = (h, m)^T$. In this model, $u = m/h$ is the horizontal velocity. For this project, we choose $g = 1$ and the spatial domain $\Omega = (0, 2)$.

The goal of this project is to implement the Finite Difference Method for the shallow water equation.

- 1.1 (a) Discretize the domain Ω with N cells. Implement the Finite Difference Method with Lax-Friedrichs flux for (1) with

- Periodic boundary conditions i.e., $\mathbf{q}_0 = \mathbf{q}_N$ and $\mathbf{q}_{N+1} = \mathbf{q}_1$,
- Open boundary conditions i.e., $\mathbf{q}_0 = \mathbf{q}_1$ and $\mathbf{q}_{N+1} = \mathbf{q}_N$.

- (b) Test your code on the problem with the initial condition

$$h(x, 0) = h_0(x) = 1 + 0.5 \sin(\pi x) \quad m(x, 0) = m_0(x) = uh_0(x), \quad (2)$$

and

$$\mathbf{S}(x, t) = \begin{bmatrix} \frac{\pi}{2} (u - 1) \cos \pi(x - t) \\ \frac{\pi}{2} \cos \pi(x - t) (-u + u^2 + gh_0(x - t)) \end{bmatrix}, \quad (3)$$

where $u = 0.25$. The exact solution to this problem is given by

$$h(x, t) = h_0(x - t) \quad m = u h. \quad (4)$$

In your computations use periodic boundary conditions and evaluate the time-step as

$$k = \text{CFL} \frac{\Delta x}{\max_i (|u_i| + \sqrt{gh_i})},$$

with $\text{CFL} = 0.5$. Plot the numerical solution at the final time $T = 0.5$. Measure the error of the scheme at $T = 2$ as a function of Δx and plot the results on a log-log graph.

- 1.2 (a) Set $\mathbf{S} = 0$ and run the program for each of the following initial conditions

$$h(x, 0) = 1 - 0.1 \sin(\pi x), \quad m(x, 0) = 0, \quad (5)$$

$$h(x, 0) = 1 - 0.2 \sin(2\pi x), \quad m(x, 0) = 0.5, \quad (6)$$

with periodic boundary conditions. Plot the solution at time $T = 0.5$. Comment on the regularity of the solution and the performance of the scheme.

- (b) For each initial function (5) and (6), measure the error of the scheme at $T = 0.5$ as a function of Δx and plot the results on a log-log graph. To measure the error, compare each numerical solution with a reference numerical solution computed on a very fine mesh.

- 1.3 (a) For $\mathbf{S} = 0$, implement the following initial condition

$$h(x, 0) = 1, \quad m(x, 0) = \begin{cases} -0.5 & x < 1 \\ 0.0 & x > 1 \end{cases}, \quad (7)$$

with open boundary conditions. Obtain a reference solution using the Lax-Friedrichs flux.

- (b) Plot the solutions with the Lax-Friedrichs scheme at time $T = 0.5$ with varying mesh size, and compare them with the reference solution. Comment on the type of waves that can be seen in the solution.
- (c) Repeat the same experiment with the Lax-Wendroff. Does the numerical solution look the same as before? Does it converge to the Lax-Friedrichs solution with mesh refinement? If not, can you explain why this is the case?