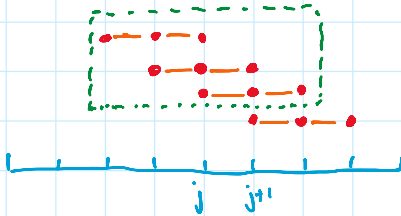


What could we improve on the ENO class of methods?

- 1) We have used $2m-1$ cell values but only get an approximation of order m $\mathcal{O}(h^m)$ even in smooth regions.



Many computational results are discarded.

- 2) The choice of stencil is sensitive to small changes in the data.

$$S = \{x_{j-1}, x_j, x_{j+1}\}$$



$$S_L = \{x_{j-2}, x_{j-1}, x_j, x_{j+1}\}$$

$$f[x_{j-2}, x_{j-1}, x_j, x_{j+1}]$$



$$S_R = \{x_{j-1}, x_j, x_{j+1}, x_{j+2}\}$$

$$f[x_{j-1}, x_j, x_{j+1}, x_{j+2}]$$

If both are very close to each other at machine precision, the choice is numerically unstable.

We expect this to happen in smooth regions.

What can we do to mitigate those problems?

- Use ENO near discontinuities
- Use a linear combination of all stencils in smooth regions.

⇒ WENO: Weighted ENO scheme

Let's follow that line of thought in more detail.

We get a new approximation of the right interface value via a "linear combination of stencil":

$$U_{j+\frac{1}{2}} = \sum_{r=r_0}^{m-1+r_0} \underline{d_r^{r_0}} \underline{U_{j+\frac{1}{2}}^{(r)}}$$

where $r_0 \in \{-1, 0\}$,

m is the stencil size,

| the values $U_{j+\frac{1}{2}}^{(r)}$ are computed with a shift r ,

| and the coefficients $d_r^{r_0}$ are yet to be determined.

Recall that r_0 determines the family of stencils:

$r_0 = 0 \rightarrow$ all stencils contain x_j

$r_0 = -1 \rightarrow$ all stencils contain x_{j+1}

We can choose the coefficients $d_r^{r_0}$ such that

$$U_{j+\frac{1}{2}} = \mathcal{U}(x_{j+\frac{1}{2}}) + \mathcal{O}(h^{2m-1})$$

over smooth regions. These coefficients can be computed from Taylor expansion

There are also called linear weights. For a uniform mesh, we can just precompute them.

For consistency, we must have

$$\sum_{r=r_0}^{m-1+r_0} d_r^{r_0} = 1, \quad d_r^{r_0} \geq 0$$

So $U_{j+\frac{1}{2}}$ is a convex combination of the stencil values

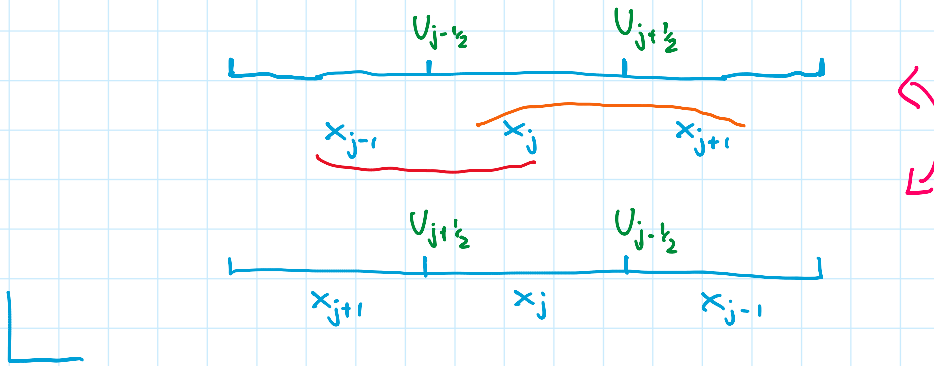
We also have approximations for the left interface value

$$U_{j-\frac{1}{2}} = \sum_{r=r_-}^{m-1+r_-} \hat{d}_r^{r_-} U_{j-\frac{1}{2}}^{(r)}$$

$$U_{j-\frac{1}{2}} = \sum_{r=r_0}^{m-1+r_0} \hat{d}_r^{r_0} U_{j-\frac{1}{2}}^{(r)}$$

where $\hat{d}_r^{r_0} = d_{m-1+r}^{r_0}$ by mirroring the coefficients.

Example: $r_0 = 0, m = 2$



This results in a linear scheme, which is problematic near discontinuities.

Whence we decide to adapt the coefficients for each cell

Consider a reconstruction of the form

$$U_{j+\frac{1}{2}} = \sum_{r=r_0}^{r_0+m-1} \omega_r^{r_0} U_{j+\frac{1}{2}}^r$$

with new coefficients that have the following properties.

1) For consistency and stability, we require that

$$\sum_{r=r_0}^{m-1+r_0} \omega_r^{r_0} = 1$$

$$\omega_r^{r_0} \geq 0$$

2) In smooth regions, to preserve the approximation quality

$$\omega_r^{r_0} = d_r^{r_0} + \mathcal{O}(h^{m-1})$$

In that case

$$\begin{aligned} U(x_{j+\frac{1}{2}}) - \sum_{r=r_0}^{m-1+r_0} \omega_r^{r_0} U_{j+\frac{1}{2}}^r \\ = \underbrace{U(x_{j+\frac{1}{2}}) - \sum_{r=r_0}^{m-1+r_0} d_r^{r_0} U_{j+\frac{1}{2}}^r}_{\mathcal{O}(h^{2m-1})} + \sum_{r=r_0}^{m-1+r_0} (d_r^{r_0} - \omega_r^{r_0}) U_{j+\frac{1}{2}}^r \end{aligned}$$

and therefore

$$\begin{aligned} \sum_{r=r_0}^{m-1+r_0} (\omega_r^{r_0} - d_r^{r_0}) U_{j+\frac{1}{2}}^r \\ = \sum_{r=r_0}^{m-1+r_0} (\omega_r^{r_0} - d_r^{r_0}) U_{j+\frac{1}{2}}^r + \underbrace{\sum_{r=r_0}^{m-1+r_0} (\omega_r^{r_0} - d_r^{r_0}) U(x_{j+\frac{1}{2}})}_{=0} \\ = \sum_{r=r_0}^{m-1+r_0} \underbrace{(\omega_r^{r_0} - d_r^{r_0})}_{\mathcal{O}(h^{m-1})} \underbrace{(U_{j+\frac{1}{2}}^r - U(x_{j+\frac{1}{2}}))}_{\mathcal{O}(h^m)} = \mathcal{O}(h^{2m-1}) \end{aligned}$$

Hence the accuracy $\mathcal{O}(h^{2m-1})$ is retained by small perturbations of the linear coefficients.

3) In non-smooth regions, we want

$$\omega_r^{(r_0)} \in \{0, 1\}$$

to effectively pick only one stencil (the smoothest one)

What are possible choices with such requirements?

We define

$$\omega_r^{r_0} = \frac{\alpha_r}{\sum \alpha_r}, \quad \alpha_r = \frac{d_r^{r_0}}{(\varepsilon + \beta_r)^{2p}}$$

where $\varepsilon > 0$, $p > 0$, and $\beta_r \geq 0$ are parameters.

Typically $\varepsilon = 10^{-6}$ and $p = 1$. Chosen adaptively

Clearly, the consistency and stability conditions are satisfied.

Parameter β_r is the smoothness indicator of the r -th stencil.

- For smooth stencils: $\beta_r = C(h) (1 + \mathcal{O}(h^{m-1}))$
↑ good ↑ independent of r
 $\mathcal{O}(h^m)$

Then $\omega_r^{r_0} \approx d_r^{r_0}$

- For non-smooth stencils: $\beta_r = \mathcal{O}(1)$
↑ bad

Then $\omega_r^{r_0} \approx \mathcal{O}(h^{2p})$

So how do we pick the smoothness indicators β_r ?

The original publications on the WENO scheme used:

$$\beta_r = \sum_{l=1}^m \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} h^{2l-1} \left(\partial_x^l \overset{\text{interpolating polynomial with shift } r}{\Pi_r} \right)^2 dx$$

where Π_r is the reconstructed polynomial for u in the r -th stencil.

Example For $m=2$, the smoothness indicators are

$$\beta_0 = (u_{j+1} - u_j)^2 \quad \beta_1 = (u_j - u_{j-1})^2$$

Example For $m=3$, the indicators are

Example for $m=3$, the indicators are

$$\beta_0 = \frac{13}{12} (u_j - 2u_{j+1} + u_{j+2})^2 + \frac{1}{4} (3u_j - 4u_{j+1} + u_{j+2})^2$$

$$\beta_1 = \frac{13}{12} (u_{j-1} - 2u_j + u_{j+1})^2 + \frac{1}{4} (u_{j-1} - u_{j+1})^2$$

$$\beta_2 = \frac{13}{12} (u_{j-2} - 2u_{j-1} + u_j)^2 + \frac{1}{4} (u_{j-2} - 4u_{j-1} + 3u_j)^2$$

Generally speaking, the terms are squares of differences of the cell values.

$$\beta_r = \sum_{\ell=1}^m \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} h^{2\ell-1} (\partial_x^\ell \pi_r)^2 dx$$

where we know that

$$\pi_r(x) = h \sum_{q=0}^{m-1} u_{q+j-r} \left(\sum_{i=q+1}^m (l_i^{(j-r-\frac{1}{2})})'(x) \right)$$

Taking derivatives, we get

$$\partial_x^\ell \pi_r(x) = h \sum_{q=0}^{m-1} u_{q+j-r} \left(\underbrace{\sum_{i=q+1}^m \partial_x^\ell (l_i^{(j-r-\frac{1}{2})})'(x)}_{Q_{q,r}^\ell(x)} \right)$$

$Q_{q,r}^\ell(x)$

We can express $Q_{q,r}^\ell(x)$ by its Taylor expansion, starting from the Taylor expansion of each Lagrange polynomial

$$l_i^{(j-r-\frac{1}{2})}(x) = \sum_{k=0}^m \frac{c_{k,i}}{k!} x^k$$

consequently

$$\partial_x^\ell l_i^{(j-r-\frac{1}{2})} = \sum_{k=0}^{m-\ell} \frac{c_{k+\ell,i}}{k!} x^k$$

$$\Rightarrow Q_{q,r}^l(x) = \sum_{k=0}^{m-l} \frac{1}{k!} \left(\sum_{i=q+1}^m c_{k+l,i} \right) x^k$$

We compute the integral

$$\int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} (\partial_x^l \pi_r)^2 dx = \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} \underbrace{\left[\sum_{q=0}^{m-1} u_{q+j-r} Q_{q,r}^l(x) \right]^2}_{u^T \cdot Q^l u} dx$$

where

$$Q_{i,j}^l = \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} Q_{i,r}^l(x) Q_{j,r}^l(x) dx$$

Hence the coefficients β_r can be computed as a quadratic form with some uniform matrix Q of the $2m-1$ values of u that are in the regions of the stencils

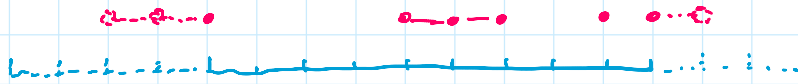
What is known theoretically?

- It is possible to set optimal order accuracy $\mathcal{O}(h^{2m-1})$ for smooth solutions
- Unfortunately, less is known for stability theory
- There are many variants
- Some variants of WENO preserve positivity
- SP-WENO: sign property, but only 3rd order

What are practical issues / concerns with ENO / WENO?

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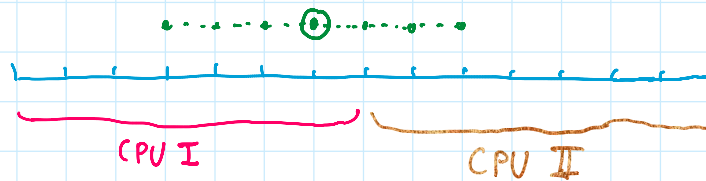
- For boundary value problems, we need to extend the computational domain via ghost nodes



Something is relevant for the finite difference for the Neumann problem

$$-u'' = f, \quad u'(0) = 0, \quad u'(1) = 0$$

- The higher the order, the larger the stencil, the less locality. That's issue with parallel computing



- The computational effort can be quite high for large m