We want to measure the smoothness of a function via discrete measurements of its point values. For that purpose we use so-called divided differences.

Suppose we are given nodal points $x_0, x_1, \dots, x_n$ with distance h.

- $f[x_i] = f(x_i)$ approximates f at x_g

  $p_0(x) = f[x_g]$

- $$f[x_i, x_{i+1}] = \frac{f(x_{i+1}) - f(x_i)}{x_{i+1} - x_i} = \frac{f[x_{i+1}] - f[x_i]}{x_{i+1} - x_i}$$

  approximates f' at x_g

- $$f[x_i, x_{i+1}, x_{i+2}] = \frac{f[x_{i+1}, x_{i+2}] - f[x_i, x_{i+1}]}{x_{i+2} - x_i}$$
  $$= \frac{1}{2} \frac{f[x_2] - 2f[x_1] + f[x_0]}{h^2}$$

  approximates $\frac{1}{2} f''$ at x_g

- More generally:
  $$f[x_j, x_{j+1}, \dots, x_{j+l}] = \frac{f[x_{j+1}, \dots, x_{j+l}] - f[x_j, \dots, x_{j+l-1}]}{x_{j+l} - x_j}$$

  approximates $\frac{1}{l!} f^{(l)}$ at x_g

Inspired by the Taylor polynomials, one may consider polynomials

$$p_0(x) = f[x_0]$$

$$p_1(x) = f[x_0] + f[x_0, x_1](x - x_0)$$

$$p_1(x) = f[x_0] + f[x_0,x_1](x-x_0)$$

$$p_2(x) = f[x_0] + f[x_0,x_1](x-x_0) + f[x_0,x_1,x_2](x-x_1)(x-x_0)$$

$$\vdots$$

(Newton interpolation)

$$p_m(x) = \sum_{\ell=0}^{m} f[x_0,\dots,x_\ell] \boxed{\prod_{i=0}^{\ell-1}(x-x_i)}$$ Newton polynomials

We are more interested in how those divided differences can measure smoothness. Conceptually, as $h \to 0$

$$f[x_0,x_1] = \frac{f(x_1)-f(x_0)}{x_1-x_0}$$

$\to$ $f'(\xi)$ if f is smooth where $\xi \in [x_0,x_1)$

$\to$ $\mathcal{O}(h^{-1})$ if f has a jump

$$f[x_0,x_1,x_2] = \frac{f(x_2)-2f(x_1)+f(x_0)}{x_2-x_0}$$

$\to$ $\frac{1}{2}f''(\xi)$ if f is smooth

$\to$ $\mathcal{O}(h^{-2})$ otherwise

$$\vdots$$

$$f[x_0,x_1,\dots,x_m]$$

$\to$ $\frac{1}{m!}f^{(m)}(\xi)$ if f is smooth

$\to$ $\mathcal{O}(h^{-m})$ otherwise

For that reason, the magnitude of the divided difference gives an idea of the smoothness of f.

Application to stencil selection

We begin with the stencil including the node x_j

$$S_0 = \{x_j\}$$

Given a stencil

$$S_\ell = \{ x_{j-p}, \dots, x_j, \dots, x_{j+q} \} \qquad p, q \geq 0$$

we consider the candidate stencils

$$S_\ell^- = \{ x_{j-p-1}, \dots, x_{j+q} \}$$
$$S_\ell^+ = \{ x_{j-p}, \dots, x_{j+q+1} \}$$

We let $S_{\ell+1} = S_\ell^-$ or $S_{\ell+1} = S_\ell^+$ depending on which stencil has the smallest divided difference.

Thus we construct a collection of stencils containing x_j.
We compute approximations to the interface values

$$U_{j+\frac{1}{2}}^- = \sum_{i=0}^{m-1} C_{i,r}^m \, \bar{U}_{j+i-r}$$

and $$U_{j-\frac{1}{2}}^+ = \sum_{i=0}^{m-1} C_{i,r-1}^m \, \bar{U}_{j+i-r}$$

We use those approximate interface values in a numerical scheme

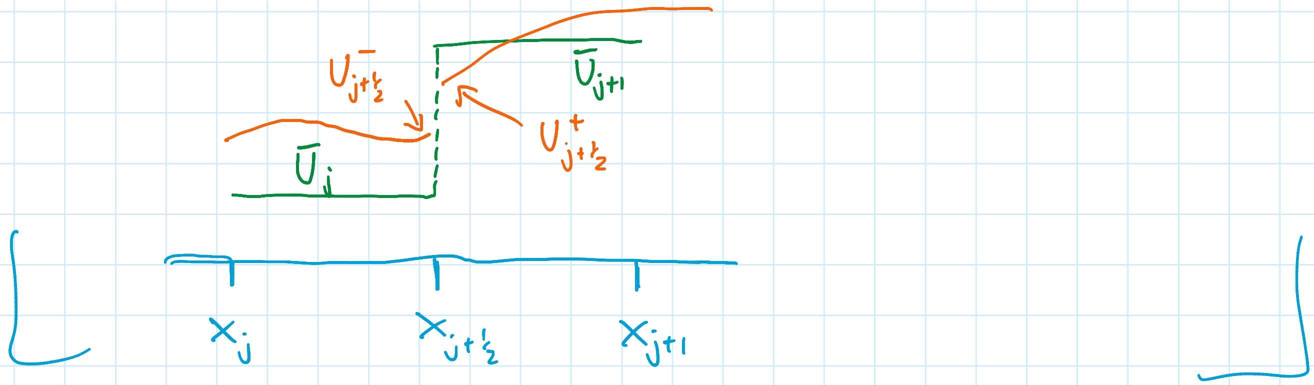
$$\bar{U}_j^{n+1} = \bar{U}_j^n - \frac{k}{h}\Big(F(U_{j+\frac{1}{2}}^-, U_{j+\frac{1}{2}}^+) - F(U_{j-\frac{1}{2}}^-, U_{j-\frac{1}{2}}^+) \Big)$$

Generally speaking, such essentially non-oscillatory schemes work well in practice, but our theoretical understanding is limited.

Nevertheless, the following property is known

We have the so-called sign property

$$\operatorname{sign}(\bar{U}_{j+1} - \bar{U}_j) = \operatorname{sign}(U_{j+\frac{1}{2}}^+ - U_{j+\frac{1}{2}}^-)$$


$U^-_{j+\frac12}$
$\bar{U}_{j+1}$
$U^+_{j+\frac12}$
$\bar{U}_j$
x_j
$x_{j+\frac12}$
x_{j+1}