How do we find A^* ???

Thm If the system is equipped with a convex entropy function $\eta(U)$, then A^* exists.

But how to compute A^* ?

a) Consider

$U : \xi \mapsto U(\xi)$

$$f(U_R) - f(U_L) = \int_0^1 \partial_\xi f(U(\xi))\, d\xi$$

$$= \int_0^1 \nabla f(U(\xi)) \cdot \partial_\xi U \, d\xi$$

where $U(0) = U_L$ and $U(1) = U_R$

We make the so-called "Roe linearization":

$$U(\xi) = U_L + \xi(U_R - U_L) \quad \Rightarrow \quad \partial_\xi U = (U_R - U_L)$$

$$f(U_R) - f(U_L) = \underbrace{A^*}_{???} \cdot (U_R - U_L)$$

a) Integrate exactly:

$$A^* := \int_0^1 \nabla f(U(\xi))\, d\xi$$

Perhaps not easy to compute :-S

b) Alternatively: (quadrature rule?)

$$A^* := \nabla f\left(\frac{U_L + U_R}{2}\right)$$

Roe condition not guaranteed :-S

$$U(\tfrac{1}{2}) = \frac{U_L + U_R}{2}$$

c) Alternatively:

$$A^* := \frac{1}{2}\left(\nabla f(U_L) + \nabla f(U_R)\right)$$

Not necessarily hyperbolic :-S

$\frac{1}{2}U(0) + \frac{1}{2}U(1)$

We reconsider the Roe condition from a different angle. The Roe condition reflects the assumption of one dominant discontinuity with speed s.

The Rankine-Hugoniot condition reads

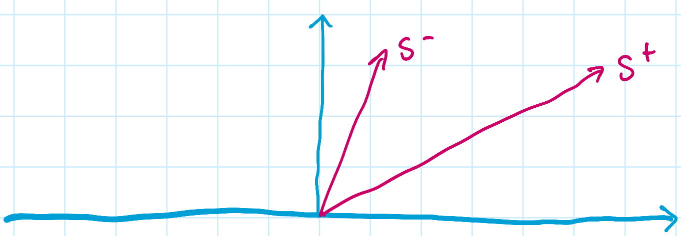
$$f(U_L) - f(U_R) = s \cdot (U_L - U_R)$$

If this is a solution to the linearized system

$$A^*(U_L - U_R) = s \cdot (U_L - U_R)$$

So the Roe condition reflects the assumption of one strong discontinuity.

We improve this by incorporating $\boxed{\text{two}}$ strong shocks



Assuming knowledge on the speeds s^- and s^+, we set

$$F(U_L, U_R) = \begin{cases} f(U_L) & \text{if } s^- > 0 \quad \text{(both +)} \\ \dfrac{s^+ f(U_L) - s^- f(U_R) + s^+ s^- (U_L - U_R)}{s^+ - s^-} & \text{if } s^- < 0 < s^+ \quad \text{(mixed)} \\ f(U_R) & \text{if } s^+ < 0 \quad \text{(both −)} \end{cases}$$

$A^*(U_L, U_R)$

This is known as HLL flux