What about nonlinear systems?

$$\partial_t U + \partial_x (f(U)) = 0$$

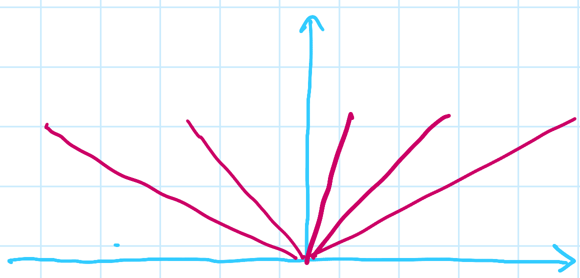
$$\Leftrightarrow \quad \partial_t U + \nabla_x f(U) \cdot \partial_x U = 0$$

$$\Leftrightarrow \quad \partial_t U + A(U) \cdot \partial_x U = 0$$

We call $A(U) = \nabla_x f(U) \in \mathbb{R}^{m \times m}$ the flux Jacobian

If the conservation law is hyperbolic, then the flux Jacobian must be diagonalizable with real eigenvalues

Riemann problem



We have m discontinuities and $m-1$ intermediate states

Since this is nonlinear, can we even find a solution, i.e., use the RH condition?

Yes if the jump is not too big

$$|U_L - U_R| < \varepsilon$$

(using linear approx. of f)

Godonov's method requires exact solutions, but that is generally impractical.

$\Downarrow$

We solve an approximate linearized problem

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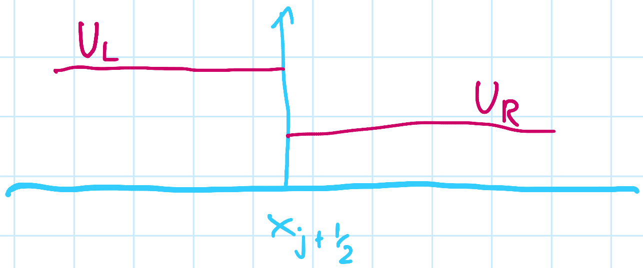


Replace $A(U)$ by $A^*(U_L, U_R)$

yet to be defined

Then we solve the local linearized problem

$$\partial_t U + A^*(U_L, U_R)\, \partial_x U = 0$$

What conditions do we impose A^* ???

1) A^* must be diagonalizable with real eigenvalues

preserve hyperbolicity

2) $A^*(U_L, U_R) \longrightarrow \nabla_U f(U)$ as $U_L, U_R \to U$

consistency

3) $A^*(U_L, U_R)\,(U_R - U_L) = f(U_R) - f(U_L)$

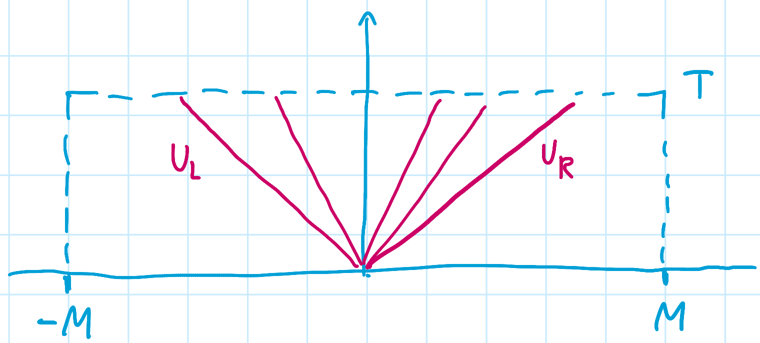
(Roe condition)

We then define the flux

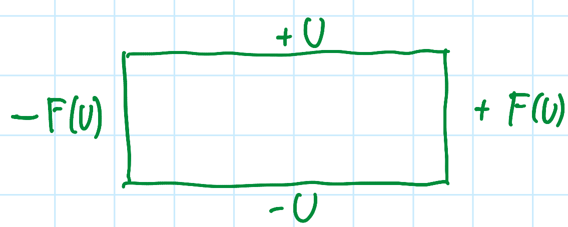
$$F_{j+\frac{1}{2}} = \frac{1}{2}\big(f(U_L) - f(U_R) \big) - \frac{1}{2}|A^*|\,(U_L - U_R)$$

The existence of such A^* will be discussed later.

Consider the (nonlinear CL) over a space-time box



$$0 = \iint \begin{pmatrix} \partial_t \\ \partial_x \end{pmatrix} \cdot \begin{pmatrix} U \\ F(U) \end{pmatrix} dt\,dx \xRightarrow[\text{Gauss}]{}$$



1) Integration + Gauss over $[-M, M] \times [0,T]$

$$\int_{-M}^{M} U(x,T) - U(x,0)\,dx = -\int_0^T f(U_R) - f(U_L)\,dt$$

We notice immediately

$$\int_{-M}^{M} U(x,T)\,dx = M(U_L + U_R) - T\big(f(U_R) - f(U_L) \big)$$

2) Integration + Gauss over $[-M,0]\times[0,T]$

$$\int_{-M}^{0} U(x,T)\,dx = M\cdot U_L - T\Big(f(U_L) - \underbrace{f(U_{j+\frac{1}{2}})}_{F_{j+\frac{1}{2}}}\Big)$$

3) Integration + Gauss over $[0,M]\times[0,T]$

$$\int_{0}^{M} U(x,T)\,dx = M\cdot U_R - T\Big(\underbrace{f(U_{j+\frac{1}{2}})}_{F_{j+\frac{1}{2}}} - f(U_R)\Big)$$

Since $U(x,t)$ is difficult to solve exactly, we replace the integrand with an approximate solution. Consider

$$\partial_t \tilde{U} + A^* \partial_x \tilde{U} = 0 \qquad A^* = A^*(U_L, U_R)$$

We repeat:

4) $$\int_{-M}^{M} \tilde{U}(x,T)\,dx = M(U_L + U_R) - T(A^* U_R - A^* U_L)$$

5) $$\int_{-M}^{0} \tilde{U}(x,T)\,dx = M\cdot U_L - T\big(A^* U_L - A^* U_{j+\frac{1}{2}}\big)$$

6) $$\int_{0}^{M} \tilde{U}(x,T)\,dx = M\cdot U_R - T\big(A^* U_{j+\frac{1}{2}} - A^* U_R\big)$$

We use

1) + 4) : $$A^*(U_R - U_L) \approx f(U_R) - f(U_L)$$

2) + 5) :

A)
$$F_{j+\frac{1}{2}} \approx f(U_L) - A^*\left(U_L - U_{j+\frac{1}{2}}(0)\right)$$
$$= f(U_L) + \sum_{\lambda_i < 0} \lambda_i^* \gamma_i^* \underline{S}_i^* \qquad \gamma_i^* = \beta_i^* - \alpha_i^*$$

using that A^* is diagonalizable with real eigenvalues

3) + 6) :

B)
$$F_{j+\frac{1}{2}} \approx f(U_R) - \sum_{\lambda_i > 0} \lambda_i^* \gamma_i^* \underline{S}_i^*$$

We use $\frac{1}{2}(A) + \frac{1}{2}(B)$

$$F_{j+\frac{1}{2}} = \frac{1}{2}\Big(f(U_L) - f(U_R)\Big) - \underbrace{\frac{1}{2}\sum_{i=1}^{m} |\lambda_i|^* \gamma_i^* S_i^*}_{\frac{1}{2}A^*(U_R - U_L)}$$