We return to our discussion of systems of conservation laws. Suppose we have a linear CL (linear system of conservation laws)

$$\partial_t \underline{U} + \underline{\underline{A}}\, \partial_x \underline{U} = 0$$

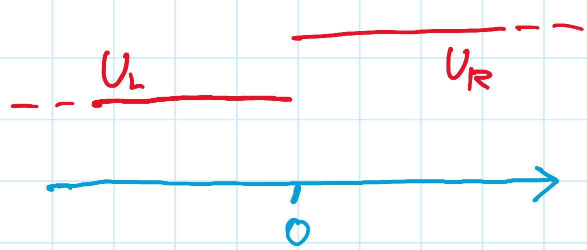
We assume the CL is strictly hyperbolic, so that A is diagonalizable with distinct eigenvalues $\lambda_1 > \lambda_2 > \dots > \lambda_m$

$$\underline{\underline{\Lambda}} = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_m \end{pmatrix} \qquad \underline{\underline{S}} = \begin{pmatrix} | & | & & | \\ s_1 & s_2 & \cdots & s_m \\ | & | & & | \end{pmatrix}$$

$$\underline{\underline{A}} = \underline{\underline{S}}\, \underline{\underline{\Lambda}}\, \underline{\underline{S}}^{-1}$$

We consider the Riemann problem of that CL with initial data

$$U_0(x) = \begin{cases} U_L & x < 0 \\ U_R & x > 0 \end{cases}$$



$$U_L = \sum_{i=1}^{m} \alpha_i s_i, \qquad U_R = \sum_{i=1}^{m} \beta_i s_i$$

$$\alpha = (\alpha_1, \alpha_2, \dots, \alpha_m) \qquad \beta = (\beta_1, \beta_2, \dots, \beta_m)$$

$$U_L = S\alpha \qquad\qquad U_R = S\beta$$

We switch to the "characteristic coordinate system" of the state variable, so that

$$V_{i,0} = \begin{cases} \alpha_i & x < 0 \\ \beta_i & x > 0 \end{cases} \qquad V_0 = \begin{pmatrix} V_{1,0} \\ V_{2,0} \\ \vdots \\ V_{m,0} \end{pmatrix}$$

The solution is given by

$$V_i\,(x,t) = \begin{cases} \alpha_i & \text{if } x < \lambda_i t \\ \beta_i & \text{if } x > \lambda_i t \end{cases} \qquad V(x,t) = \begin{pmatrix} V_1 \\ V_2 \\ \vdots \\ V_m \end{pmatrix}$$

$\boxed{\text{Example}}$ We consider the matrix

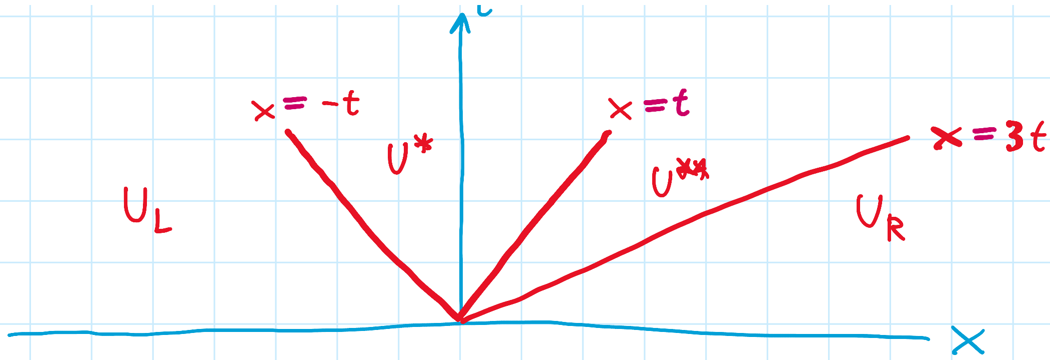
$$\underline{\underline{A}} = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix} \qquad \lambda_1 = 3, \quad \lambda_2 = 1, \quad \lambda_3 = -1$$

$$S_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \qquad S_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \qquad S_3 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

Consider the case

$$U_L = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \quad U_R = \begin{pmatrix} 0 \\ 0 \\ -2 \end{pmatrix} \quad \Rightarrow \quad \alpha = \begin{pmatrix} \sqrt{2} \\ 1 \\ \sqrt{2} \end{pmatrix}, \quad \beta = \begin{pmatrix} \sqrt{2} \\ 1 \\ -\sqrt{2} \end{pmatrix}$$

The solution can be depicted as follows



We have
$$U_L = \alpha_1 S_1 + \alpha_2 S_2 + \alpha_3 S_3$$
$$U^* = \alpha_1 S_1 + \alpha_2 S_2 + \beta_3 S_3$$
$$U^{**} = \alpha_1 S_1 + \beta_2 S_2 + \beta_3 S_3$$
$$U_R = \beta_1 S_1 + \beta_2 S_2 + \beta_3 S_3$$

As a rule of thumb, we have $m-1$ intermediate states

The solution will generally have m discontinuities and $m-1$ intermediate states

The jump across the p-th discontinuity

$$[u]_p = (\beta_p - \alpha_p)\, S_p \qquad 1 \le p \le m$$

$$[f]_p = \underline{\underline{A}}\, [u]_p = (\beta_p - \alpha_p)\, \underline{\underline{A}}\, \underline{S}_p$$
$$= (\beta_p - \alpha_p)\, \lambda_p\, \underline{S}_p = \lambda_p [u]_p$$

Thus $\quad [f]_p = \lambda_p [u]_p \qquad$ (RH condition)

We see

$$\underline{U}_L - \underline{U}_R = U_L - U^* + U^* - U^{**} + \dots - U_R$$

$$\underline{U}_L - \underline{U}_R = \underbrace{\underline{U}_L - \underline{U}^*}_{[U]_m} + \underbrace{\underline{U}^* - \underline{U}^{**}}_{[U]_{m-1}} + \dots - \underline{U}_R$$

$$= [U]_1 + [U]_2 + \dots + [U]_m$$

$$= \sum_{p=1}^{m} [U]_p$$

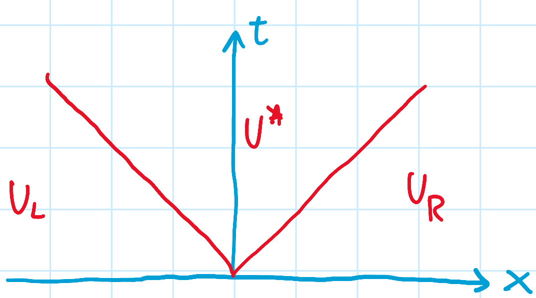
$$= \sum_{p=1}^{m} (\beta_p - \alpha_p)\, \underline{S}_p = \underline{\underline{S}}\,(\beta - \alpha)$$

So we can describe the solution to the Riemann problem

1. Find eigenvalues and -vectors
2. Assemble intermediate states by formula above

$\boxed{\text{Example}}$ $\quad \underline{\underline{A}} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ $\quad \lambda_1 = 1 \quad \lambda_2 = -1$

$$S_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad S_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$



$$A\left(U^* - U_L\right) = -\left(U^* - U_L\right) \quad (1)$$

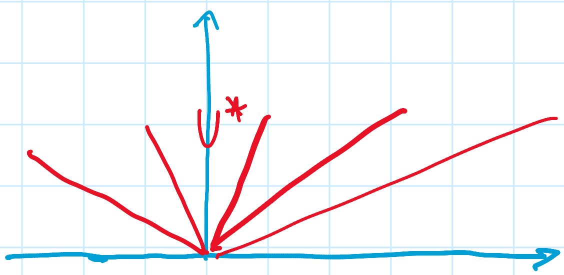
$$A\left(U_R - U^*\right) = +\left(U_R - U^*\right) \quad (2)$$

$$(1) + (2) \Rightarrow \quad A\,(U_R - U_L) = U_L + U_R - 2U^*$$

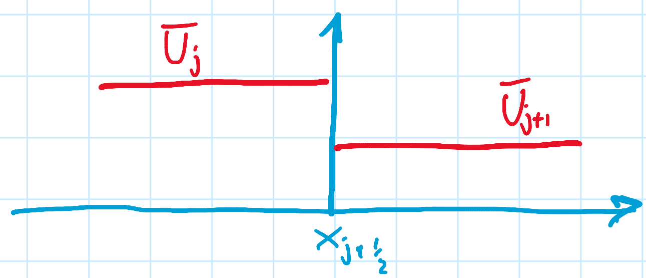
$$\Rightarrow \quad U^* = \tfrac{1}{2}(U_R + U_L) - \tfrac{1}{2}A\,(U_R - U_L)$$

Let us revisit the flux in the finite volume method

$$\partial_t U + \underline{\underline{A}}\, \partial_x U = 0$$



What is $F = f(U^*)$



$$\underline{\overline{U}}_j = \underline{\underline{S}}\, \underline{\alpha} \qquad \underline{\overline{U}}_{j+1} = \underline{\underline{S}}\, \underline{\beta}$$

$$\lambda_1, \dots, \lambda_q > 0$$
$$\lambda_{q+1}, \dots, \lambda_m < 0$$

We find that

$$\underline{U}(x,t) = \sum_{i=1}^{q} \alpha_i \underline{S}_i + \sum_{i=q+1}^{m} \beta_i \underline{S}_i$$

$$= \underline{\overline{U}}_j + \sum_{i=q+1}^{m} \underbrace{(\beta_i - \alpha_i)}_{\gamma_i} \underline{S}_i$$

$$= \underline{\overline{U}}_{j+1} - \sum_{i=1}^{q} (\beta_i - \alpha_i) \underline{S}_i$$

Now we see

$$\underline{U}(x_{j+\frac{1}{2}}, t) = \underline{\overline{U}}_j + \sum_{i:\, \lambda_i < 0} \gamma_i \underline{S}_i$$

$$= \overline{U}_{j+1} - \sum \gamma_i S_i$$

$$= \underline{\bar{U}}_{j+1} - \sum_{i:\,\lambda_i > 0} \gamma_i \underline{S}_i$$

The flux becomes

$$F_{j+\frac{1}{2}} = \underline{\underline{A}}\, U^* = \underline{\underline{A}}\, \underline{\bar{U}}_j + \sum_{i:\,\lambda_i < 0} \gamma_i \lambda_i \underline{S}_i$$

$$\underline{\underline{A}}\, \underline{\bar{U}}_{j+1} - \sum_{i:\,\lambda_i > 0} \gamma_i \lambda_i \underline{S}_i$$

Yet another approach

$$\underline{\underline{A}}^{\pm} = \underline{\underline{S}}\, \underline{\underline{\Lambda}}^{\pm}\, \underline{\underline{S}}^{-1} \quad \text{where}$$

$$\Lambda^{+}_{ii} = \max(0, \lambda_i) \qquad \Lambda^{-}_{ii} = \min(0, \lambda_i)$$

So we have $\Lambda = \Lambda^{+} + \Lambda^{-}$

$$\Lambda^{+} = \begin{pmatrix} \lambda_1 & & & & & \\ & \ddots & & & & \\ & & \lambda_q & & & \\ & & & 0 & & \\ & & & & \ddots & \\ & & & & & 0 \end{pmatrix} \qquad \Lambda^{-} = \begin{pmatrix} 0 & & & & & \\ & \ddots & & & & \\ & & 0 & & & \\ & & & \lambda_{q+1} & & \\ & & & & \ddots & \\ & & & & & \lambda_m \end{pmatrix}$$

Since $\underline{\bar{U}}_{j+1} - \underline{\bar{U}}_j = \sum_{i=1}^{m} \gamma_i \underline{S}_i$

we have $\underline{\underline{A}}\left(\underline{\bar{U}}_{j+1} - \underline{\bar{U}}_j\right) = \sum_{i=1}^{m} \gamma_i \lambda_i \underline{S}_i$

We summarize

$$F_{j+\frac{1}{2}} = \underline{\underline{A}}\, \bar{U}_j + A^{-}\left(\bar{U}_{j+1} - \bar{U}_j\right) \qquad (3)$$

$$F_{j+\frac{1}{2}} = \underline{\underline{A}}\,\bar{U}_j + A^-(\bar{U}_{j+1} - \bar{U}_j) \qquad (3)$$

$$= \underline{\underline{A}}\,\bar{U}_{j+1} - A^+(\bar{U}_{j+1} - \bar{U}_j) \qquad (4)$$

Average (3) and (4) :

$$F_{j+\frac{1}{2}} = \frac{1}{2}\underline{\underline{A}}(U_j + U_{j+1}) + \frac{1}{2}(A^- - A^+)(\bar{U}_{j+1} - \bar{U}_j)$$

$$= \frac{1}{2}\underline{\underline{A}}(U_j + U_{j+1}) - \frac{1}{2}|A|(U_{j+1} - U_j)$$

where $|A| = \underline{\underline{S}}\,|\Lambda|\,\underline{\underline{S}}^{-1}$

$$|\Lambda| = \begin{pmatrix} |\lambda_1| & & & \\ & |\lambda_2| & & \\ & & \ddots & \\ & & & |\lambda_m| \end{pmatrix}$$

We observe that

$$f(U) = \underline{\underline{A}} \cdot U \qquad \|\underline{\underline{A}}\|_2 \leq |\lambda_1|$$

We can define

$$F_{j+\frac{1}{2}} := \frac{f(\bar{U}_j) + f(\bar{U}_{j+1})}{2} - \frac{1}{2}|\lambda_1|(\bar{U}_{j+1} - \bar{U}_j)$$

$\Longrightarrow$ Lax - Friedrich