# 18 Gudonov scheme

We recall that the cell averages and flux averages

$$\bar{U}_j^n = \frac{1}{h}\int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} u(x, t^n)\, dx$$

$$\bar{F}_{j+\frac{1}{2}}^n = \frac{1}{k}\int_{t^n}^{t^{n+1}} f(x_{j+\frac{1}{2}}, t)\, dt$$

satisfy the "numerical" formula

$$\bar{U}_j^{n+1} = \bar{U}_j^n + \frac{k}{h}\left( \bar{F}_{j+\frac{1}{2}}^n - \bar{F}_{j-\frac{1}{2}}^n \right)$$

No numerical approximation yet whatsoever. Generally not computable.

But you use this as a starting point to develop a numerical scheme, as we replace exact quantities by approximations.

$\boxed{\#1}$ Suppose we have $\bar{U}_j^n$. We define a polynomial $p_j$ of degree $p+q$ by setting

$$\frac{1}{h}\int_{x_{\ell-\frac{1}{2}}}^{x_{\ell+\frac{1}{2}}} p_j(x)\, dx = \bar{U}_\ell^n \qquad \ell = j-p, \dots, j+q$$

$x_{\ell-\frac{1}{2}}$

Most simple choice: $\quad p = q = 0$

$$p_j = \bar{U}_j^n \qquad \ell = j$$

$\boxed{\#2}$ We define the Riemann problem

$$\partial_t u + \partial_x F(u) = 0$$

$$\ast \qquad u_0(x) = \begin{cases} p_{j-1}(x_{j-\frac{1}{2}}) & x < x_{j-\frac{1}{2}} \\ p_j(x_{j-\frac{1}{2}}) & x > x_{j-\frac{1}{2}} \end{cases}$$

$\boxed{\#3}$ Whatever emerges from the discontinuity at $x_{j-\frac{1}{2}}$ will not reach the neighboring interval boundaries if

$$k\,|f'| < h \qquad \frac{k}{h}|f'| < 1$$

$\boxed{\#4}$ Solve the local Riemann problem **exactly** and integrate exactly

$$\bar{F}^n_{j+\frac{1}{2}} := \int_{t^n}^{t^n+k} f(\, u(x_{j+\frac{1}{2}}, t)\, )\, dt$$

Thus we can perform the time-step.

:-) Only approximation at the very beginning

:-( Not computable unless in special cases

A compromise is to solve + integrate approximately.