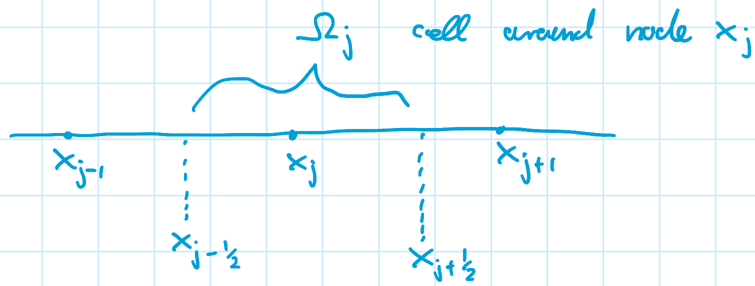
17 Finite Volume Schemes

The discussion of finite difference schemes started from the differential form. Let us start instead from the integral form.



Integral form

$$\int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} \left[u(x, t^{n+1}) - u(x, t^n) \right] dx = - \int_{t^n}^{t^{n+1}} f(u(x_{j+\frac{1}{2}}, t)) - f(u(x_{j-\frac{1}{2}}, t)) \, dt$$

Define cell averages

$$\bar{U}_j^n = \frac{1}{h} \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} u(x, t^n) \, dx$$

and a numerical flux

$$F_{j+\frac{1}{2}}^n = \frac{1}{k} \int_{t^n}^{t^{n+1}} f(u(x_{j+\frac{1}{2}}, t)) \, dt$$

Then we have

$$\bar{U}_j^{n+1} = \bar{U}_j^n - \frac{k}{h} \left(F_{j+\frac{1}{2}}^n - F_{j-\frac{1}{2}}^n \right)$$

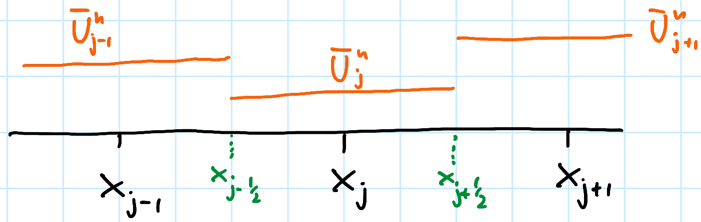
This is the prototype for a numerical method. Instead of approximating point values, we approximate cell averages.

We don't have made any numerical approximation yet. The above formula will be used as a blueprint to build numerical methods. The key idea is that we approximate the integrals.

$$F^n_{j+\frac{1}{2}} = \frac{1}{k}\int_{t_n}^{t_{n+1}} f\left(u(x_{j+\frac{1}{2}}, t)\right) dt$$

Godunov scheme

$\boxed{\text{Idea}}$ We start with piecewise constant states in each cell



- Close to the cell interfaces, we have Riemann problems.
- The solution will be, for a short time period, constant away from the edge. We approximate the numerical flux by taking the value at the edge after that short time.