Let us focus on monotone schemes and the iterates produced by such schemes

$$U_j^{n+1} = G(U^n)_j = U_j^n - \frac{k}{h}\left(F_{j+\frac{1}{2}}^n - F_{j-\frac{1}{2}}^n \right)$$

What is the connection to entropy solutions?

We introduce the notion of Kružkov entropy pair:

$$\eta(u) = |u-c|, \qquad \Psi(u) = \operatorname{sign}(u-c)\Big(f(u) - f(c) \Big), \qquad c \in \mathbb{R}$$

This is an entropy pair.

<u>Thm</u> If U^n is obtained by a monotone scheme satisfies the entropy condition

$$\frac{\eta(U_j^{n+1}) - \eta(U_j^n)}{k} + \frac{\Psi_{j+\frac{1}{2}}^n - \Psi_{j-\frac{1}{2}}^n}{h} = 0$$

for every Kružkov entropy pair.

What are the limitations of monotone schemes?

$$x_j = h \cdot j, \qquad j = 0, 1, \dots, N$$

$$t^n = k \cdot n \qquad n = 0, 1, \dots, K \qquad K \cdot k = T$$

Suppose that U^n and u are the numerical and exact solutions, respectively. What about:

Suppose that U and u are the numerical and exact solutions, respectively. What about:

$$\| u(x_j, t^n) - U_j^n \| \xrightarrow[h,k \downarrow 0]{} 0$$

We plug in the exact solutions into the scheme

$$u(t^n)_j := u(x_j, t^n)$$

$$u(t^{n+1})_j = G(u(t^n))_j + k \underbrace{T_j^n}_{\text{truncation error}}$$

We want the truncation error to vanish as $h, k \to 0$.

1. Assume that u exists and is smooth enough
2. T_j^n is obtained via Taylor expansion.

<u>Example</u> <u>Transport equation</u>

$$u_t + c\, u_x = 0, \qquad c > 0$$

$\boxed{\text{FTBS}}$ $$U_j^{n+1} = G(U^n)_j = U_j^n - \frac{k}{h} c\left(U_j^n - U_{j-1}^n \right)$$

$$u(x_j, t^{n+1}) = u(x_j, t^n) + k\, \partial_t u(x_j, t^n) + \frac{k}{2} \partial_{tt}^2 u(x_j, t^n) + O(k^3)$$

$$u(x_{j-1}, t^n) = u(x_j, t^n) - h\, \partial_x u(x_j, t^n) + \frac{h^2}{2} \partial_{xx}^2 u(x_j, t^n) + O(h^3)$$

$$T_j^n = \frac{1}{k}\Big(u(t^{n+1}, x_j) - G(u(x_j, t^n)) \Big)$$

$$= \frac{1}{k}\Big(u + k\partial_t u + \frac{k}{2}\partial_{tt}^2 u + \mathcal{O}(k^3)$$

$$- u + \frac{k}{h}c\big(u - u + h\partial_x u + \frac{h^2}{2}\partial_{xx} u + \mathcal{O}(h^3)\big)\Big)$$

$$= \frac{1}{k}\Big(\underbrace{k(\partial_t u + c\,\partial_x u)}_{=0} + \frac{k^2}{2}\partial_{tt} u + khc\,\partial_{xx}^2 u + \mathcal{O}(k^3 + h^3)\Big)$$

$$= \frac{k}{2}\partial_{tt}^2 u + ch\,\partial_{xx}^2 u + \text{H.O.T.}$$

$$=: \mathcal{O}(h, k)$$

The truncation goes to zero if k and h go the zero provided the true solution is regular enough.

Exercise:

FTFS $\quad \mathcal{O}(h, k)$

FTCS $\quad \mathcal{O}(h^2, k)$

We have seen that this is not enough in practice, since FTFS and FTCS quickly blow up

We define the local solution error

$$\varepsilon_j^n := u(x_j, t^n) - U_j^n$$

We have

$$U_j^{n+1} = G(U^n)_j\,, \qquad u(t^{n+1}, x_j) = G(u(t^n))_j + k\,T_j^n$$

For simplicity, we assume that $G(u)$ is linear in u

$$\varepsilon_j^0 = u(0, x_j) - U_j^0 = 0$$

$$\varepsilon_j^1 = G(\varepsilon^0)_j + k\,T_j^0$$

$$\varepsilon_j^2 = G(\varepsilon^1)_j + k\,T_j^1$$

$$= G^2(\varepsilon^0)_j + k\left(G(T^0)_j + T_j^1\right)$$

$$\vdots$$

$$\varepsilon_j^n = G^n(\varepsilon_j^0) + k\sum_{i=0}^{n-1} G^{n-i-1}(T^i)_j$$

Define discrete p-norms

$$\|U\|_{h,p} = \left(h\sum_j |U_j|^p\right)^{1/p}$$

$$\|U\|_{h,\infty} = \max_j |U_j|$$

The global solution error satisfies

$$\|\varepsilon^n\|_{h,p} \le \|G^n(\varepsilon^0)\|_{h,p} + k\sum_{i=0}^{n-1}\|G^{n-i-1}(T^i)\|_{h,p}$$

$$\le |||G^n|||_{h,p}\,\|\varepsilon^0\|_{h,p} + \underbrace{kn}_{t^n}\,\max_i\left(|||G^{n-i-1}|||_{h,p}\cdot\|T^i\|_{h,p}\right)$$

Consequently, the total error goes to zero if

1) $\|\varepsilon^0\|_{h,p} \longrightarrow 0$ for $h \to 0$ $\qquad |||G|||_{h,p} = \sup_{x\in\mathbb{R}^{N+1}} \frac{\|Gx\|_{h,p}}{\dots}$

1) $\|\varepsilon^0\|_{h,p} \longrightarrow 0$ for $h \to 0$

$$\|G\|_{h,p} = \sup_{x \in \mathbb{R}^{N+1}} \frac{\|Gx\|_{h,p}}{\|x\|_{h,p}}$$

2) $\|T^i\|_{h,p} \longrightarrow 0$ for $h \to 0$

1)+2) are called consistency of the scheme
The order of the scheme is (p,q) if

$$\|T^i\|_{h,s} = \mathcal{O}(h^p, k^q) = \mathcal{O}(h^p) + \mathcal{O}(k^q)$$

3) $\|G^n\|_{h,p} \leq C$ (Stability)

A sufficient condition for stability is

$$\|G\|_{h,p} \leq 1 + \alpha k$$

$$\Rightarrow \|G^n\|_{h,p} \leq \|G\|^n_{h,p} \leq (1+\alpha k)^n$$

$$\leq e^{\alpha k n} \leq e^{\alpha T} =: C$$

Stability measures sensitivity in initial conditions.

What about the stability of monotone schemes?

$$U_j^{n+1} = G(U^n)_j, \qquad u(t^{n+1})_j = G(u(t^n))_j + kT_j^n$$

$$\Rightarrow \varepsilon_j^{n+1} = G(u(t^n))_j - G(U^n)_j + kT_j^n$$

$$\Rightarrow \|\varepsilon_j^{n+1}\|_1 \leq \|G(u(t^n))_j - G(U^n)_j\|_1 + k\|T_j^n\|_1$$

L^1 contraction

$$\leq \|u(t^n)_j - U_j^n\|_1 + k\|T_j^n\|_1$$

$$\leq \|\varepsilon_j^n\|_1 + k\|T_j^n\|_1$$

Recall

$$\| U_j \|_1 + k \| T_j \|_1$$

Recall

$$\| U^n_j \|_1 = \sum_j | U^n_j |, \qquad \| U^n_j \|_{h,1} = h \, \| U^n_j \|_1$$

Hence

$$\| \varepsilon^{n+1}_j \|_1 \leq \| \varepsilon^n_j \|_1 + k \, \| T^n_j \|_1$$

$$\Rightarrow \quad \| \varepsilon^{n+1}_j \|_{h,1} \leq \| \varepsilon^n_j \|_{h,1} + k \, \| T^n_j \|_{h,1}$$

Thus, monotone schemes are stable