Some fluxes

- Lax-Friedrichs Flux ✓

$$F_{LF}(u,v) = \frac{f(u)+f(v)}{2} - \frac{\alpha}{2}(v-u)$$

where $\alpha = \max_u |f'(u)|$

- Lax-Wendroff Flux

$$F_{LW}(u,v) = \frac{f(u)+f(v)}{2} - \frac{k}{2h} f'\left(\frac{u+v}{2}\right)\Big(f(v)-f(u)\Big)$$

- Roe Flux

$$F_{Roe}(u,v) = \frac{f(u)+f(v)}{2} - \frac{1}{2}\left|\frac{f(v)-f(u)}{v-u}\right|(v-u)$$

$$= \begin{cases} f(u) & \text{if } s>0 \\ f(v) & \text{if } s<0 \end{cases} \qquad s = \frac{f(v)-f(u)}{v-u}$$

- All three fluxes are consistent and Lipschitz

It seems that is not enough.

Let us focus on the case of a finite difference scheme in where the numerical fluxes only depend on the immediate neighbors.

$$U_j^{n+1} = U_j^n - \frac{k}{h}\left(F^n_{j+\frac{1}{2}} - F^n_{j-\frac{1}{2}}\right)$$

$$= U_j^n - \frac{k}{h}\Big(F(U_j^n, U_{j+1}^n) - F(U_{j-1}^n, U_j^n)\Big)$$

$$=: G(U_{j-1}^n, U_j^n, U_{j+1}^n)$$

We also write with abuse of notation

$$G(U^n)_j = G(U_{j-1}^n, U_j^n, U_{j+1}^n)$$

We call the scheme monotone if G is increasing in each argument. Let's express that in terms of the differentials.

$$0 \leq \frac{\partial G}{\partial U_{j-1}^n} = \frac{k}{h} \partial_1 F(U_{j-1}^n, U_j^n)$$

$$0 \leq \frac{\partial G}{\partial U_j^n} = 1 - \frac{k}{h}\Big(\partial_1 F(U_j^n, U_{j+1}^n) - \partial_2 F(U_{j-1}^n, U_j^n) \Big)$$

$$0 \leq \frac{\partial G}{\partial U_{j+1}^n} = -\frac{k}{h} \partial_2 F(U_j^n, U_{j+1}^n)$$

The first and the third inequality are equivalent to F increasing in its first argument and decreasing in its second argument.

The second inequality requires that the quotient k/h must be sufficiently small, that is, the time step must be smaller than the space step.

Example Lax-Friedrichs Flux

$$F_{LF}(u,v) = \tfrac{1}{2}\big(f(u)+f(v)\big) - \frac{\alpha}{2}(v-u)$$

We find that $\left(\alpha = \max_{u\in\mathbb{R}} |f'|\right)$

$$\partial_u F(u,v) = \tfrac{1}{2} f'(u) + \frac{\alpha}{2} \geq 0$$

$$\partial_v F(u,v) = \tfrac{1}{2} f'(v) - \frac{\alpha}{2} \leq 0$$

Finally

$$1 - \frac{k}{h}\left(\frac{f'(U_j^n)}{2} + \frac{\alpha}{2} - \frac{f'(U_j^n)}{2} + \frac{\alpha}{2} \right)$$

$$= 1 - \frac{k}{h}\alpha$$

Thus we need $1 - \frac{k}{h}\alpha \geq 0$

$$\iff \frac{k}{h} \leq \frac{1}{\alpha} \iff k \leq \frac{h}{\alpha}$$

$h \qquad \alpha \qquad \qquad \alpha$

The higher the derivatives of f, the smaller the time step, to have a monotone scheme.

Example Lax–Wendroff

$$F_{LW}(u,v) = \frac{1}{2}\big(f(u)+f(v)\big) - \frac{k}{2h} f'\left(\frac{u+v}{2}\right)\big(f(v)-f(u)\big)$$

Let's pick $f(u) = u^2$ and see what happens

$$F_{LW}(u,v) = \frac{1}{2}u^2 + \frac{1}{2}v^2 - \frac{k}{2h}(u+v)(u^2-v^2)$$

$$= \frac{1}{2}u^2 + \frac{1}{2}v^2 - \frac{k}{2h}\left(u^3 - uv^2 + u^2 v - v^3\right)$$

$$\partial_v F_{LW}(u,v) = u - \frac{k}{2h}\left(3u^2 - v^2 + 2uv\right)$$

$$= \cdots + v^2 - 2uv$$

For certain values of u and v, we cannot control the sign of the derivative

Exercise: Roe scheme is not monotone.

A few observations regarding monotone schemes.

1. If $U_j \leq V_j$ entrywise, then
$$G(U)_j \leq G(V)_j$$

2. Maximum principle
$$\min_i U_i \leq G(U)_j \leq \max_i U_i$$

3. L^1 contraction property

$$\| G(u) - G(v) \|_1 \leq \| u - v \|_1$$

4. For the iterates we have

$$TV(U^{n+1}) \leq TV(U^n) \qquad TV(U^n) = \sum_j |U^n_{j+1} - U^n_j|$$

"total variation diminishing" (TVD)

Monotone $\Rightarrow$ L^1 contraction $\Rightarrow$ TVD