What about convergence of finite difference schemes?

$$\frac{U_j^{n+1} - U_j^n}{k} + \frac{1}{h}\left(F^n_{j+\frac{1}{2}} - F^n_{j-\frac{1}{2}} \right) = 0$$

Let $\phi \in C^1_c$ and $\phi_j^n = \phi(x_j, t^n)$.
We multiply the discrete formula and sum

$$\sum_{j,n} \phi_j^n \left(\frac{U_j^{n+1} - U_j^n}{h} \right) = - \sum_{j,n} \phi_j^n \frac{1}{h}\left(F^n_{j+\frac{1}{2}} - F^n_{j-\frac{1}{2}} \right)$$

We apply "discrete integration by parts"

$$\sum_{k=0}^{p} a^k (b^{k+1} - b^k) = -a^0 b^0 - \sum_{k=1}^{p} b^k (a^k - a^{k-1}) + a^p b^{p+1}$$

and use compactness of ϕ

$$\Rightarrow \; -\frac{1}{k} \sum_{j,n} U_j^n \left(\phi_j^n - \phi_{j+1}^n \right) - \frac{1}{k} \sum_j \phi_j^0 U_j^0$$
$$= \sum_{j,n} F^n_{j+\frac{1}{2}} \frac{1}{h} \left(\phi^n_{j+1} - \phi_j^n \right)$$

We assume that the discrete solution U converge to some u in the following sense

$$\lim_{\ell \to \infty} \sum_{j,n} | U_j^n - u(x_j, t^n) | \, h_\ell k_\ell = 0$$

as $h_\ell, k_\ell \longrightarrow 0$ for $\ell \to 0$.

Then one can show that

$$\lim_{\ell \to \infty} h_\ell k_\ell \sum_{j,n} U_j^n \frac{1}{k_\ell} (\phi_j^n - \phi_j^{n-1}) = \iint_{\Omega_t \times \Omega_x} u \, \partial_t \phi \, dx dt$$

$$\lim_{\ell \to \infty} h_\ell \sum_j \phi_j^0 U_j^0 = \int_{\Omega_x} u_0(x) \phi(x, 0) \, dx$$

Assume $TV(U^n) = \sup_h \sum_j | U^n_{j+1} - U^n_j | \leq R$

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If F is Lipschitz

$$\Rightarrow \lim_{\ell \to \infty} h_\ell k_\ell \sum_{j,n} F(U^n_j, U^n_{j+1}) \frac{1}{h_\ell} (\phi^n_{j+1} - \phi^n_j) = \iint_{\Omega_x \times \Omega_t} f(v) \frac{\partial \phi}{\partial x} \, dx dt$$

That means U^n_j converges to a weak solution of the conservation law

<u>Thm</u> (Lax-Wendroff)

Suppose we have a conservative scheme with a consistent and Lipschitz flux F. If U^n_j converges to a TV bounded function u, then u is a weak solution.