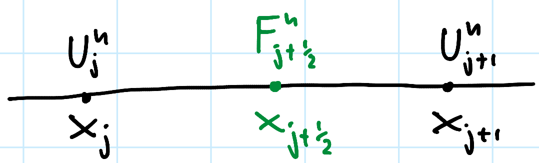
10 Finite Differences 3

General (finite difference) schemes for conservation laws have the form

$$\frac{U_j^{n+1} - U_j^n}{k} + \frac{1}{h}\left(F^n_{j+\frac12} - F^n_{j-\frac12} \right) = 0$$

$$\partial_t u \qquad\qquad \partial_x F(u)$$

Conceptually, we define the flux at nodes inbetween the nodes for u:



Update formula:

$$U_j^{n+1} = U_j^n - \frac{k}{h}\left(F^n_{j+\frac12} - F^n_{j-\frac12} \right)$$

The numerical flux $F^n_{j+\frac12}$ depends on the numerical solution U^n near the node $x_{j+\frac12}$

$$F^n_{j+\frac12} = F\Big(\underbrace{U^n_{j-p}, \dots, U^n_{j-1}, U^n_j, U^n_{j+1}, \dots, U^n_{j+q}}_{p+1+q \text{ entries of } U^n} \Big)$$

Example

$$F^n_{j+\frac12} = F(U_j^n) = (U_j^n)^2 \qquad \text{Burgers}$$

$$F^n_{j+\frac12} = F(U_j^n) = c\, U_j^n \qquad \text{Transport}$$

Important properties of numerical fluxes:

1) consistent flux: $F(U, U, \dots, U) = f(U)$

ensures locally constant solution stays constant

2) first-order approximation: there exists $M > 0$ s.t.

$$\left| F(U_1, U_2, \dots, U_{p+q+1}) - f(u) \right| \leq M \cdot \max_{1 \leq i \leq p+q+1} |U_i - u|$$

Remark: In practice, we only use the direct neighbours of U_j^n, that is, U_{j-1}^n and U_{j+1}^n; so $p, q \leq 1$

Intuitively, the "integral" of the numerical solution should be conserved. We compute

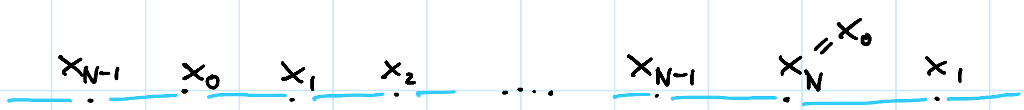
$$\sum_j U_j^{n+1} = \sum_j U_j^n - \frac{k}{h} \sum_j F_{j+\frac{1}{2}}^n - F_{j-\frac{1}{2}}^n \qquad (1)$$

We use a telescoping sum

$$\sum_j F_{j+\frac{1}{2}}^n - F_{j-\frac{1}{2}}^n$$

$$= F_{\frac{1}{2}}^n - F_{-\frac{1}{2}}^n + F_{\frac{3}{2}}^n - F_{\frac{1}{2}}^n + \dots + F_{N-\frac{1}{2}} - F_{N-\frac{3}{2}}$$

$$= F_{N-\frac{1}{2}}^n - F_{-\frac{1}{2}}^n$$



If we assume periodic boundary conditions, then the system is "physically closed" and the sum $\sum_j U_j^n$ is the same at every time step. In particular, if

$$h \sum_j U_j^n = \int_{\Omega_x} u_0(x)\, dx$$

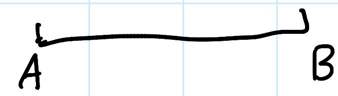
then the integral is preserved.

If instead we assume that the solution / flux is constant

If instead we assume that the solution / flux is constant outside of the interval, then (1) becomes

$$\sum_j U_j^{n+1} = \sum_j U_j^n - F^n_{N-\frac{1}{2}} + F^n_{-\frac{1}{2}}$$

$$= \sum_j U_j^n - f(U_B) + f(U_A)$$



<u>Thm</u> A scheme in conservative form

$$U_j^{n+1} = U_j^n + \frac{k}{h}\left(F^n_{j+\frac{1}{2}} - F^n_{j-\frac{1}{2}} \right)$$

with a consistent numerical flux is conservative at the discrete level.

In such a scheme, shocks will move according to the RH conditions.