07 Entropy Pairs

Suppose that f is the flux function of some conservation law.

An entropy pair is a pair of functions $(\eta(u), \Psi(u))$ where

- η is strictly convex
- We have $\Psi'(u) = \eta'(u)\, f'(u)$

"compatibility condition"

<u>Observation 1</u> If u is smooth, then

$$\begin{aligned} 0 &= \eta'(u)\left(\partial_t u + \partial_x f(u)\right) \\ &= \eta'(u)\,\partial_t u + \eta'(u)\, f'(u)\,\partial_x u \\ &= \partial_t \eta + \partial_x \Psi \end{aligned}$$

<u>Observation 2</u> If u is not smooth, we consider the limit in vanishing viscosity

$$\partial_t u_\varepsilon + \partial_x f(u_\varepsilon) = \varepsilon\, \partial^2_{xx} u_\varepsilon$$

We multiply by $\eta'(u_\varepsilon)$

$$\eta'(u_\varepsilon)\,\partial_t u_\varepsilon + \eta'(u_\varepsilon)\,\partial_x f(u_\varepsilon) = \varepsilon\, \eta'(u_\varepsilon)\,\partial^2_{xx} u_\varepsilon$$

$$\Longrightarrow \quad \partial_t \eta(u_\varepsilon) + \partial_x \Psi(u_\varepsilon) = \varepsilon\, \eta'(u_\varepsilon)\,\partial^2_{xx} u_\varepsilon$$

$$\Longrightarrow \quad \partial_t \eta(u_\varepsilon) + \partial_x \Psi(u_\varepsilon) = \varepsilon\, \eta'(u_\varepsilon)\, \partial_{xx}^2 u_\varepsilon$$

We observe

$$\eta'(u_\varepsilon)\, \partial_{xx}^2 u_\varepsilon = \partial_x\big(\eta'(u_\varepsilon)\, \partial_x u_\varepsilon\big) - \underbrace{\eta''(u_\varepsilon)}_{\geq 0}\, \underbrace{(\partial_x u_\varepsilon)^2}_{\geq 0}$$

$$\eta'(u_\varepsilon)\, \partial_{xx}^2 u_\varepsilon \leq \partial_x\big(\eta'(u_\varepsilon)\, \partial_x u_\varepsilon\big)$$

Now we integrate over space and time:

$$\iint \partial_t \eta(u_\varepsilon) + \partial_x \Psi(u_\varepsilon)\, dx\, dt$$

$$\leq \iint \varepsilon\, \partial_x\big(\eta'(u_\varepsilon)\, \partial_x u_\varepsilon\big)\, dx\, dt$$

When we work over an interval $[x_1, x_2]$ up to time $T > 0$, then the last integral is

$$\varepsilon \int_0^T \int_{x_1}^{x_2} \partial_x\big(\eta'(u_\varepsilon)\, \partial_x u_\varepsilon\big)\, dx\, dt$$

$$= \int_0^T \varepsilon\, \big(\eta'(u_\varepsilon)\, \partial_x u_\varepsilon\big)\Big|_{x_1}^{x_2} dt \xrightarrow[\varepsilon \to 0]{} 0$$

Hence we conclude that

$$\boxed{\iint \partial_t \eta(u) + \partial_x \Psi(u)\, dx\, dt \leq 0}$$

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$$\partial_t \eta(u) + \partial_x \psi(u) \leq 0$$

Theorem If u is a weak solution to the conservation law with flux f, and (η, ψ) is any entropy pair, such that

$$\partial_t \eta(u) + \partial_x \psi(u) \leq 0,$$

then u is a weak entropy solution.