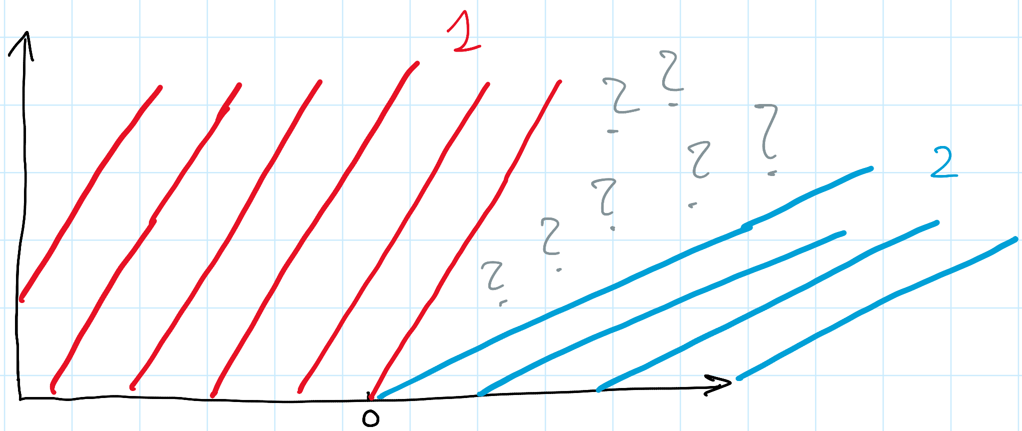
05 Examples for weak solutions

As an instructive example we consider, again, Burgers's equation.
The initial values are piecewise constant with a discontinuity.

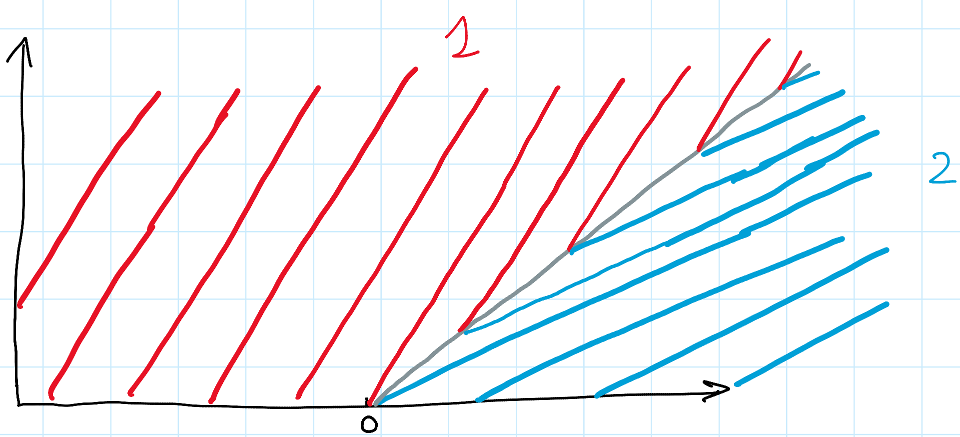
$$\partial_t u + \partial_x(u^2) = 0 \qquad u_0(x) = \begin{cases} 1 & x < 0 \\ 2 & x > 0 \end{cases}$$

We use characteristics to find a solution



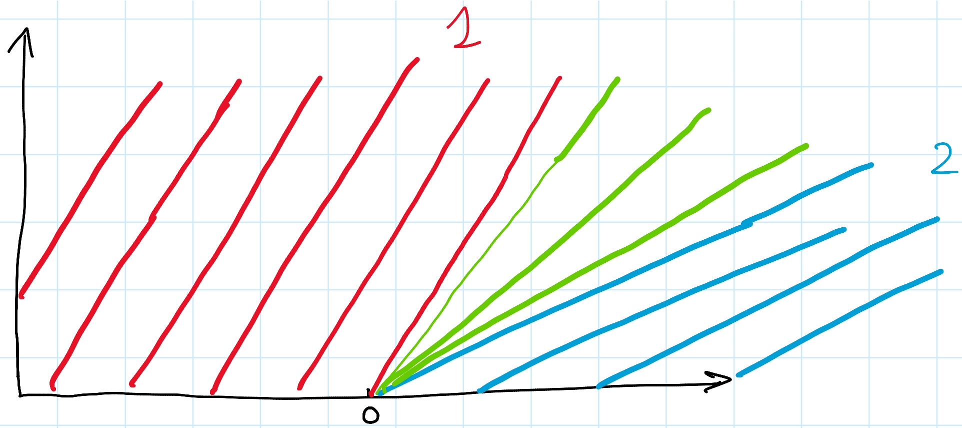
Solution 1: If there is a discontinuity along a straight line, then it has the speed

$$s = \frac{4 - 1}{2 - 1} = 3$$



$$u(x,t) = \begin{cases} 1 & x \leq st \\ 2 & x \geq st \end{cases}$$

Solution 2



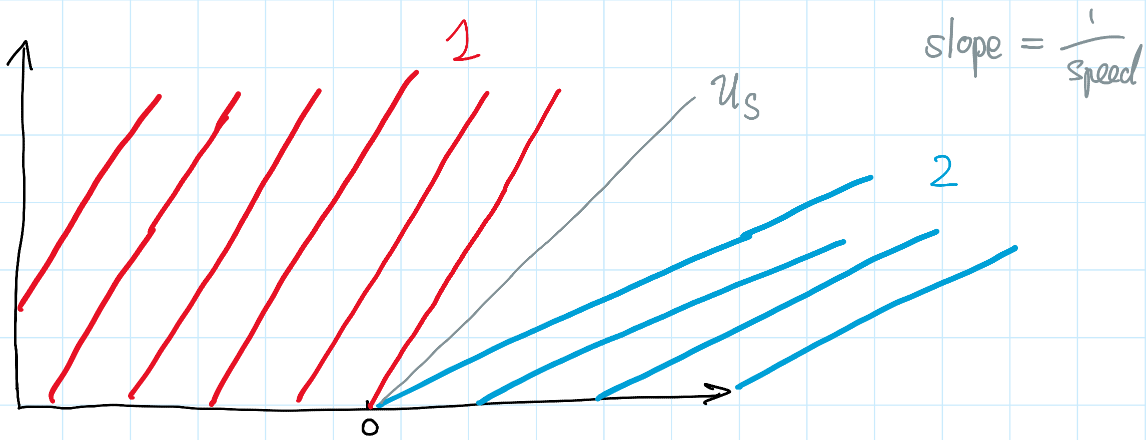
$$u(x,t) = \begin{cases} 1 & x \le 2t \\ x/2t & 2t \le x < 4t \\ 2 & x \ge 4t \end{cases}$$

for $t > 0$

$$= \begin{cases} 1 & x/t \le 2 \\ \frac{1}{2} x/t & 2 \le x/t \le 4 \\ 2 & 4 < x/t \end{cases}$$

So we have multiple weak solutions. How do we pick the one that's physical?

We want the characteristics to run into the shock.



We need $s_\ell \ge s \ge s_r$ for the slope.
For the value u we need

$$\frac{f(u) - f(u_\ell)}{u - u_\ell} \geq s \geq \frac{f(u) - f(u_r)}{u - u_r}$$

whenever u is between u_ℓ and u_r.

This is the $\boxed{\text{Lax entropy condition}}$

Thm Let $f: \mathbb{R} \to \mathbb{R}$ be strictly convex. Suppose that u and v are piecewise smooth weak solutions to

$$\partial_t u + \partial_x f(u) = 0, \qquad u(x,0) = u_0(x)$$

$$\partial_t v + \partial_x f(v) = 0, \qquad v(x,0) = v_0(x)$$

such that all discontinuities are shocks, that is, satisfy the entropy condition. Then

$$\partial_t \| u(t) - v(t) \|_{L^1} \leq 0$$

In particular, if $u_0 = v_0$, then the solution is unique.