03 Examples and Challenges

The preceding example had the form

$$\partial_t u + \partial_x (a(x,t)\, u) = 0$$

$$\partial_t u + \partial_t f(u, x, t) = 0$$

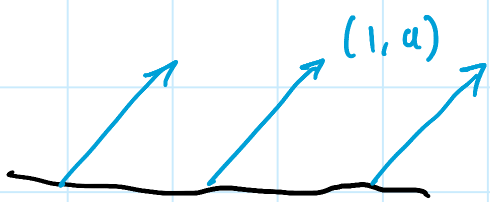
$$f(u,x,t) = a(x,t)\, u$$

Let's investigate some special cases.

<u>a = constant</u>

The PDE is

$$\partial_t u + a\, \partial_x u = 0$$



We see that the solution u must be constant in the space-time direction $(a, 1)$ so with initial values $g(x)$ at $t = 0$ we get

$$u(x,t) = g(x - a\,t)$$

<u>a = diff'able function in x</u>

The PDE is

$$\partial_t u + \partial_x \big(a(x)\, u \big) = 0$$

We take the derivative

$$\partial_x \big(a(x)\, u \big) = \partial_x a \cdot u + a(x)\, \partial_x u$$

Hence the PDE can be rewritten

$$\partial_t u + a(x)\, \partial_x u = -\partial_x a \cdot u$$

Suppose we follow the lines defined by

$$\partial_t x = a(x)$$

Then

$$\partial_t \Big(u(x(t), t) \Big)$$

$$= \partial_x u\, (x(t), t)\, a(x) + \partial_t u(x(t), t)$$

$$= -\partial_x a \cdot u(x(t), t)$$

Hence solving the ODE

$$\partial_t v(t) = -\partial_x a \cdot v(t), \qquad v(0) = u_0(x_0)$$

provides the values $v(t) = u(x(t), t)$ along the lines $x(t)$.

If the function $a(x)$ is everywhere positive/negative, then the characteristics don't intersect.

Burgers' Equation

$$\partial_t u + \partial_x (u^2) = 0$$

Assuming diff'able:

$$\partial_t u + 2u \cdot \partial_x u = 0$$

We observe that the characteristics are straight along which the value of u is constant. However, the slope of the characteristics depends on the value of u along that line, that is, the initial value $u_0(x_0)$.

So from any initial value $u_0(x)$ we propagate the value $u_0(x_0)$ along the line $x_0 + t \cdot u_0(x_0)$.

Apparently, the characteristics may intersect.

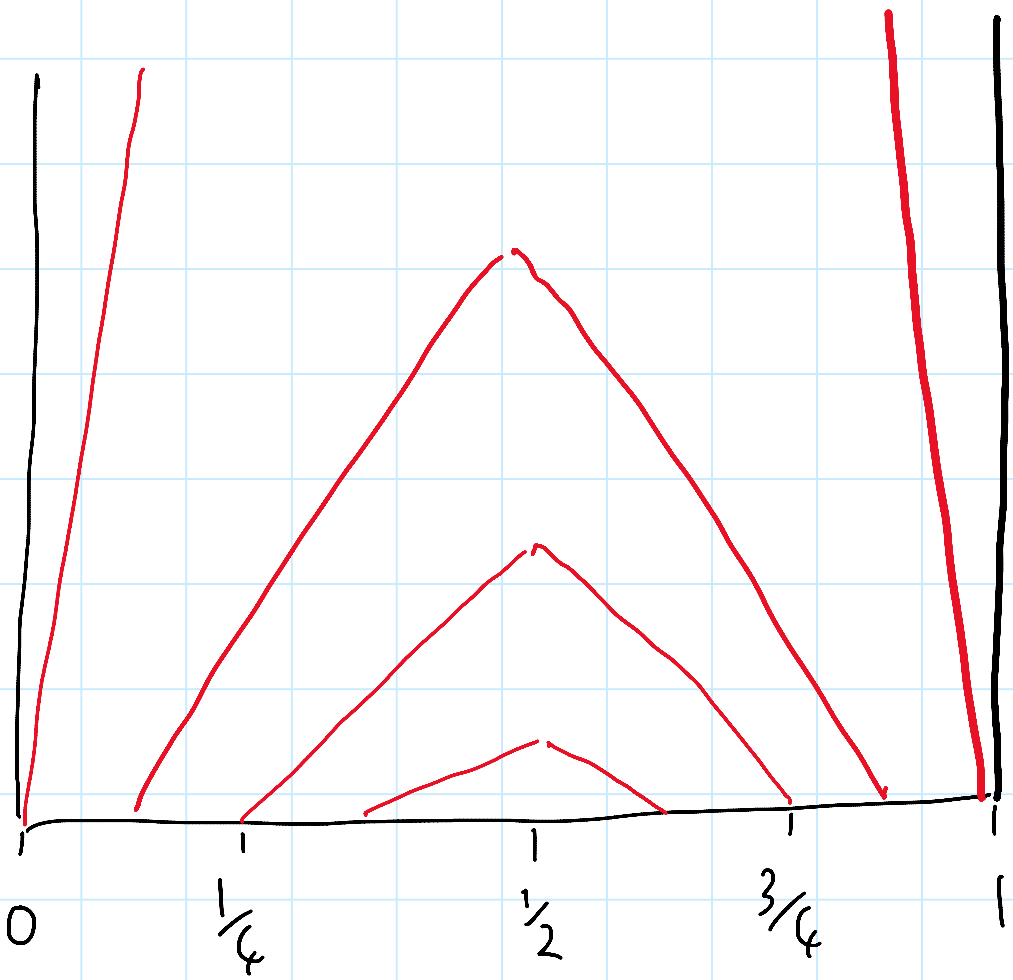
Remark:
Over an interval $[0,1]$ with periodic boundary conditions, that is, $u(0,t) = u(1,t)$ we observe that the integral of the solution is conserved:

$$\partial_t \int_0^1 u\, dx = \int_0^1 \partial_t u\, dx = \int_0^1 \partial_x u^2\, dx = u(1)^2 - u(0)^2 = 0$$

Example

We review Burgers' equation with periodic BC and smooth initial data $u_0(x) = sin(2\pi x)$

The characteristics start intersecting right after the beginning of the evolution



Generally speaking, the upper and the lower wave move into each other. The discontinuity is largest when the values 1 and -1 meet, then the discontinuity slowly dies off.

A number of observations are in order:

1. Even smooth initial data may lead to discontinuities
2. "Solutions" are non differentiable, so a new definition of solution is due.
3. Non-reversible evolution

Background of Burgers' equation

Consider the one-dimensional Navier-Stokes equation with incompressible fluid, no external force, constant pressure, and constant density

$$\partial_t u + 2u \cdot \partial_x u = \nu \Delta u$$

If the viscosity is negligible, $\nu \ll 1$, then

$$\partial_t u + 2\, u \cdot \partial_x u \approx 0$$