01 Conservation Laws

For the purpose of this class, a conservation law is a partial differential equation (PDE) of the form

$$\underbrace{\partial_t}_{\text{time-derivative}} u + \underbrace{\operatorname{div}}_{\text{(spatial) divergence}} \underbrace{f(u,x,t)}_{\text{flux (vector field)}} = 0$$

This is a first-order evolution problem.

Generally speaking, this class studies:

- Existence & Uniqueness (weak solutions, entry conditions)
- Qualitative properties (shock waves, physics background)
- Numerical methods (Finite Diff., Finite Volume, DG methods, …)

General remarks

- This is an evolution problem, i.e., time-dependent, hence we need initial data at t=0
- We know initial data and the flux, and we want to know u=u(x,t)
- More generally, the flux may be dependent on the time, that is,

  f(u) = f(u,x,t)
- u can be a scalar (scalar conservation law) or may have multiple components (system of conservation laws)

- If u is a scalar function

$$u : \mathbb{R}^d \times \mathbb{R} \longrightarrow \mathbb{R}$$

then the flux must be

$$f : \mathbb{R} \times \mathbb{R}^d \times \mathbb{R} \longrightarrow \mathbb{R}^d$$

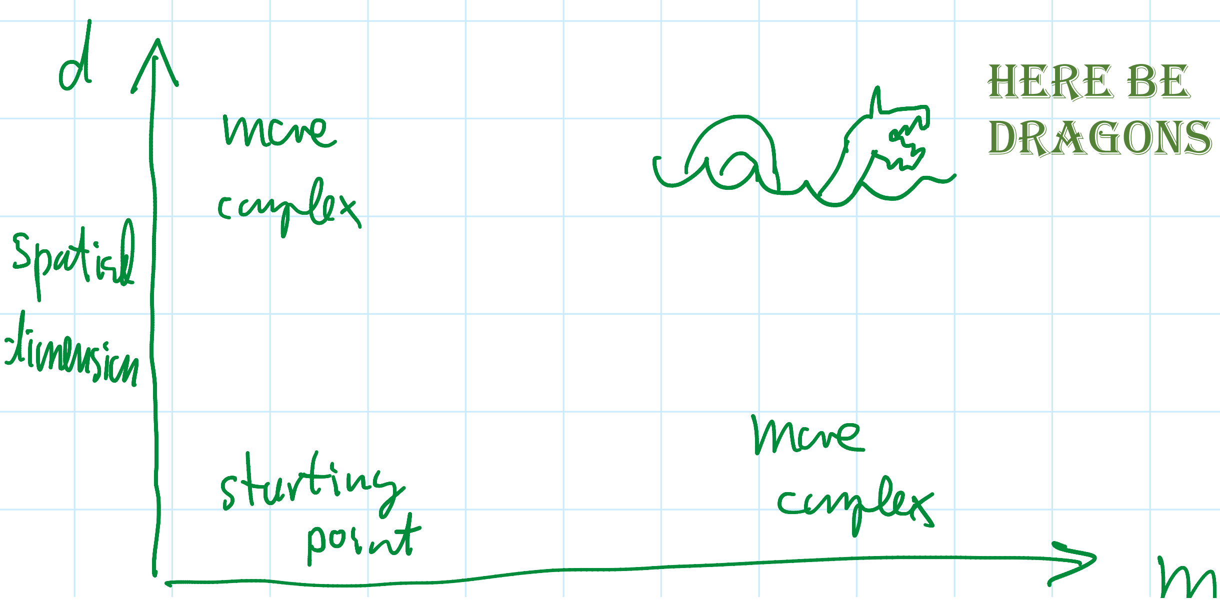
- If u has multiple (m) components

$$u : \mathbb{R}^d \times \mathbb{R} \longrightarrow \mathbb{R}^m$$

then the flux must be

$$f : \mathbb{R}^m \times \mathbb{R}^d \times \mathbb{R} \longrightarrow \mathbb{R}^{m \times d}$$

In practice, we look at a spatial domain $\Omega \subseteq \mathbb{R}^d$ and an interval $\Omega_t = [0,T]$.



$\longrightarrow m$

\# of components

Why "conservation law"?

Let $\omega \subseteq \mathbb{R}^d$ be a small domain and let $u(x,t)$ be the density of some quantity

$$\int_\omega u(x,t)\,dx = \text{amount of quantity in } \omega \text{ at time } t$$

How does this change over time?

$$\partial_t \int_\omega u(x,t)\,dx = ?$$

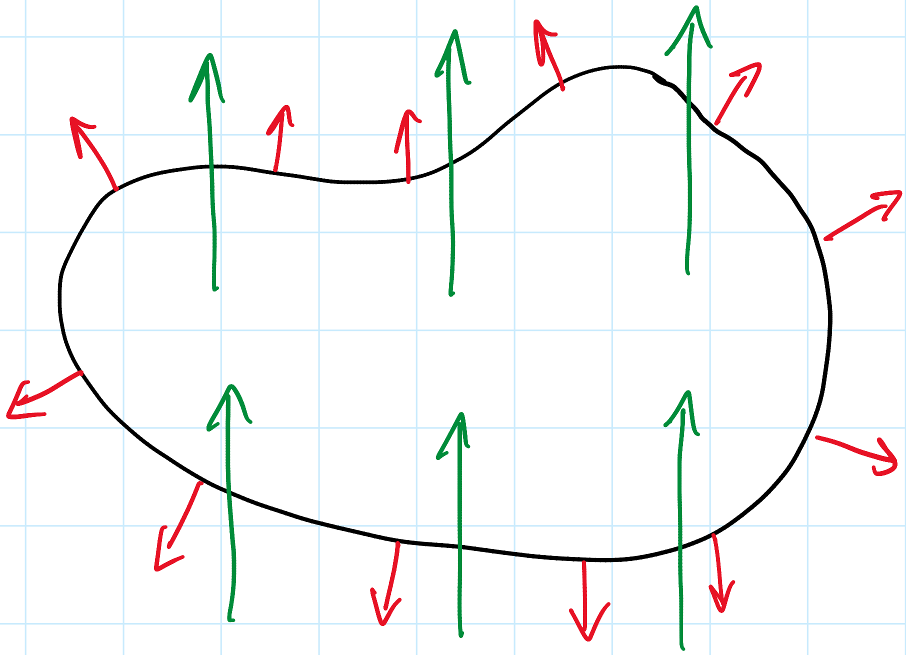
We use the conservation law

$$\partial_t \int_\omega u(x,t)\,dx = \int_\omega \partial_t u(x,t)$$

$$= - \int_\omega \operatorname{div} f(u)\,dx$$

$$= -\oint_{\omega} f(u) \cdot \vec{n} \, ds$$

outward unit normal



In practice, the balance equation

$$\partial_t \int_{\omega} u(x,t) \, dx = -\oint f(u(x,t)) \, \vec{n} \, ds$$

leads to the PDE.