Characteristics and Weak solutions

Exercise 1
Consider the partial differential equation

$$\partial_t u(x,t) + x\partial_x u(x,t) = 0.$$

Find curves $x(t)$ along which any solution must be constant. Show that every $(x,t) \in \mathbb{R} \times \mathbb{R}$ lies on such a curve. Find a solution for the initial values $u_0(x) = \cos(x)$.

Exercise 2
Suppose that u is the density of a quantity over the real line. We assume that its velocity over the real line is given by

$$v(x,t) = u(x,t)^3.$$

Describe the change of the integral of u over a subinterval $[x_1, x_2]$ in terms of the velocity v and u itself. Use the fundamental theorem of calculus to derive a conservation law, and find its characteristics. Discuss the similarity to Burgers' equation, and how this example can be generalized.

Exercise 3
An example for a nonlinear wave equation is

$$\partial_{tt} E = \partial_x \left(|\partial_x E|^{p-2} \partial_x E \right)$$

where $p \geq 1$. Reformulate this as a system of conservation laws.

Exercise 4
Consider the transport equation

$$\partial_t u(x,t) + \partial_x \left(a(t) u(x,t) \right) = 0$$

where the velocity depends only one the time variable t. Find the characteristics and show that the characteristics do not intersect.

Suppose that u describes the density of cars along a one-dimensional street. What sort of traffic flow does this conservation law describe? *This simple PDE is only for practice purposes.*

Exercise 5
(Hard) Consider the conservation law (transport equation)

$$\partial_t u(x,t) + \partial_x \left(a(x) u(x,t) \right) = 0$$

We have seen that the characteristics satisfy the following ordinary differential equation with initial values:

$$\partial_t x(t) = a(x(t)), \quad x(0) = x_0.$$

Show that the characteristics do not intersect if the velocity is always positive. [1]

[1] This is often noted casually in the literature, but can be proven rigorously.