Exercise Set 9: Linear Systems

Exercise 1
This exercise pertains to *Godunov's method* for linear systems with constant coefficients,

$$q_t + Aq_x = 0 \ . \tag{1}$$

On linear systems, Godunov's method reduces to a generalization of the upwind method where the numerical flux is given by the following equivalent expressions

$$\begin{aligned} F\left(Q_l, Q_r\right) &= AQ_l + A^-\left(Q_r - Q_l\right) & (2)\\ &= AQ_r - A^+\left(Q_r - Q_l\right) = \frac{1}{2}A\left(Q_r + Q_l\right) - \frac{1}{2}|A|\left(Q_r - Q_l\right) \ . & (3) \end{aligned}$$

Where $|A| = A^+ - A^-$ and $A^\pm = S\Lambda^\pm S^{-1}$. Here Λ^+ and Λ^- are diagonal matrices with non-negative and non-positive entries, respectively, such that $S^{-1}AS = \Lambda$ is the spectral decomposition of A, with $\Lambda = \Lambda^+ + \Lambda^-$. Especially, notice that

$$A = A^+ + A^- \ . \tag{4}$$

Consider the one dimensional acoustics equation

$$\begin{pmatrix} p \\ v \end{pmatrix}_t + \begin{pmatrix} u_0 & K_0 \\ 1/\rho_0 & u_0 \end{pmatrix} \begin{pmatrix} p \\ v \end{pmatrix}_x = 0 \ . \tag{5}$$

This system is derived from the nonlinear Euler equation by linearizing around some fixed state, as sound waves are small perturbation in a background media. Here, K_0 is the compressibility modulus, and u_0 and ρ_0 are the the velocity and pressure, respectively. The speed of sound in the medium is given by

$$c_0 = \sqrt{K_0/\rho_0} \ . \tag{6}$$

1. For (5), calculate A^+ and A^-.
2. What is the CFL condition of Godunov's method for (5)?
3. Implement Godunov's method for (5) the following two sets of initial data

$$p(x,0) = \sin\left(2\pi x\right) \ , \quad v(x,0) = 0, \quad \text{with periodic BC} \tag{7}$$

$$p(x,0) = \begin{cases} 0 & x < 0 \\ 1 & x > 0 \end{cases} \ , \quad v(x,0) = 0, , \quad \text{with open BC.} \tag{8}$$

   Use $u_0 = 1/2$, $K_0 = 1$, $p_0 = 1$ and solve on the interval $x \in [-1,1]$ with $h = 0.01$ to $T = 0.4$ and an appropriate time-step satisfying the CFL condition.

4. In the solution driven by (8), are the discontinuities visible in the numerical solution at $T = 0.4$? Plot and compare with the exact solution at the final time.
5. Now, run your code with the initial data

$$p(x,0) = \begin{cases} 1 & x < 0 \\ \sin\left(2\pi x\right) & x > 0 \end{cases} \ , \quad v(x,0) = 0 \ . \tag{9}$$

   Does the exact solution preserve the discontinuities present in the initial condition? Are you able to observe the discontinuities in the numerical solution at $T = 0.4$?