Exercise Set 7: Incremental form and Harten's lemma

Information
Consider the conservation law

$$u_t + f(u)_x = 0, \tag{1}$$

and the numerical scheme

$$u_i^{n+1} = G(u_{i-k}^n, ..., u_{i+k}^n). \tag{2}$$

We say that the scheme (2) can be put in *incremental form* if there exists two incremental coefficients $C_{i+\frac{1}{2}} = C(u_{i-k+1}^n, ..., u_{i+k}^n)$ and $D_{i+\frac{1}{2}} = D(u_{i-k+1}^n, ..., u_{i+k}^n)$, which can be used to re-write the scheme as

$$u_i^{n+1} = u_i^n - C_{i-\frac{1}{2}}\Delta^- u_i^n + D_{i+\frac{1}{2}}\Delta^+ u_i^n, \tag{3}$$

where $\Delta^+ u_i = u_{i+1} - u_i$ and $\Delta^- u_i = u_i - u_{i-1}$. Harten's lemma states that a scheme written in incremental form is TVD if i) $C_{i+\frac{1}{2}} \geq 0$, ii) $D_{i+\frac{1}{2}} \geq 0$ and iii) $C_{i+\frac{1}{2}} + D_{i+\frac{1}{2}} \leq 1$.

Exercise 1
Prove that any 3-point consistent, conservative scheme with numerical flux $F_{i+\frac{1}{2}}$ admits an incremental form with coefficients

$$C_{i+\frac{1}{2}} = \frac{k}{h}\left(\frac{f(u_{i+1}) - F_{i+\frac{1}{2}}}{\Delta^+ u_i}\right), \qquad D_{i+\frac{1}{2}} = \frac{k}{h}\left(\frac{f(u_i) - F_{i+\frac{1}{2}}}{\Delta^+ u_i}\right).$$

Exercise 2
Consider a conservative scheme with

- Lax-Friedrich flux:
$$F^{LF}(u,v) = \frac{1}{2}\left(f(u) + f(v) - \frac{h}{k}(v-u)\right),$$
- Local Lax-Friedrich/ Rusanov flux:
$$F^{LLF}(u,v) = \frac{1}{2}\left(f(u) + f(v) - \alpha(v-u)\right), \quad \alpha = \max_u |f'(u)|,$$
- Lax-Wendroff flux:
$$F^{LW}(u,v) = \frac{1}{2}\left(f(u) + f(v) - \frac{k}{h}f'\left(\frac{u+v}{2}\right)(f(v) - f(u))\right),$$
- Roe flux:
$$F^{Roe}(u,v) = \frac{1}{2}\left(f(u) + f(v) - \alpha(v-u)\right), \quad \alpha = \left|\frac{f(v) - f(u)}{v-u}\right|.$$

1. Find the incremental coefficients for each flux.
2. Check whether all three conditions of Harten's lemma are satisfied with each flux.
3. Can you say whether the numerical solution obtained with a TVD scheme is guaranteed to converge to an entropy solution?

Exercise 3
Let $f(u) = cu$. Consider a scheme with the hybrid flux,

$$F(u,v) = \theta F^{LW} + (1-\theta)F^{LF}, \quad 0 \leq \theta \leq 1,$$

which is nothing but a convex combination of the Lax-Friedrich and Lax-Wendroff fluxes. Assuming the usual CFL condition, can you find a θ that will lead to a TVD scheme?