Exercise Set 5: Conservative Methods

Exercise 1
Answer the following questions:

1. When is a numerical method said to be conservative?
2. What is the benefit of using a conservative numerical method? (Hint: Take a look at the Lax-Wendroff Theorem)
3. For a given conservation law and a conservative scheme, are we guaranteed that the weak solution obtained satisfies the entropy condition?

Exercise 2
Consider the conservative scheme with the *Engquist-Osher* flux

$$F^{EO}(u,v) = \frac{1}{2}\left(f(u) + f(v) - \int_u^v |f'(\xi)| d\xi \right). \tag{1}$$

1. Show that the numerical flux leads to a monotone scheme under suitable CFL conditions.
2. Assuming f is convex with a single minima at ω, show that the Engquist-Osher flux reduces to

$$F^{EO}(u,v) = f(\max(u,\omega)) + f(\min(v,\omega)) - f(\omega). \tag{2}$$

Exercise 3
In the previous exercises, we have seen that difficulties can arise when trying to approximate solution for linear problem. Additional issues can arise when dealing with non-linear problems. Consider the Burgers equation in the quasilinear form

$$u_t + uu_x = 0 \; . \tag{3}$$

A "natural" finite difference method can be obtained with a minor modification of the upwind method applied to the advection equation, assuming $v_j^n \geq 0$ for all j, n:

$$v_j^{n+1} = v_j^n - \frac{k}{h} v_j^n \left(v_j^n - v_{j-1}^n \right). \tag{4}$$

This method converges on smooth solutions.

1. Compute the numerical solution obtained by (4), driven by the initial condition

$$u(x,0) = \begin{cases} 1 & x < 0 \\ 0 & x \geq 0 \end{cases} . \tag{5}$$

   Implement the method in Matlab and solve (3) up to $T = 0.5$ in the interval $(-1,1)$ with initial condition (5). In your computations use $k = 0.5h$, $h = 0.01$.

2. Is the solution obtained a weak solution? Is it the entropy solution?
3. Now use the following initial condition in your code

$$u\,(x,0) = \begin{cases} 1.2 & x < 0 \\ 0.4 & x \geq 0 \end{cases} . \tag{6}$$

4. Compare the solution to the known entropy solution for Burgers, which can be constructed by considering the characteristics and shocks.

Exercise 4 1. Apply the generalization of the Lax-Friedrichs method

$$v_j^{n+1} = \frac{1}{2}\left(v_{j+1}^n + v_{j-1}^n\right) - \frac{k}{2h}\left(f\left(v_{j+1}^n\right) - f\left(v_{j-1}^n\right)\right) \tag{7}$$

to Burgers equation in conservation form,

$$u_t + \left(\frac{1}{2}u^2\right)_x = 0 \ , \tag{8}$$

with the initial conditions (5) and (6).

2. Does the numerical solutions converge to a weak solution?

3. Is this the entropy solution?

4. Show that the generalization of the Lax-Friedrichs method to nonlinear conservation laws can be written in conservative form.

Exercise 5 1. Solve the Burgers equation with the Engquist-Osher flux and the initial conditions (5) and (6).

2. Does the solution converge to the entropy solution?

3. How does the solution compare to that obtained with the Lax-Friedrichs scheme?