Exercise Set 3: Characteristics, Entropy and Weak solutions

Exercise 1 (Entropy Solutions)
Consider the conservation law

$$\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = 0, \quad u(x,0) = u_0(x), \tag{1}$$

and $f(u) = u^2/2$. Consider the following initial conditions

$$\text{(a)} \quad u_0(x) = \begin{cases} 1, & \text{for} \quad x < -1, \\ 0, & \text{for} \quad -1 < x < 1, \\ -1, & \text{for} \quad x > 1 \end{cases} \qquad \text{and} \quad \text{(b)} \quad u_0(x) = \begin{cases} -1, & \text{for} \quad x < -1, \\ 0, & \text{for} \quad -1 < x < 1, \\ 1, & \text{for} \quad x > 1. \end{cases} \tag{2}$$

Draw the profile of $u_0(x)$ and sketch the characteristics of the entropy solution $u(x,t)$ in the $x-t$ plane, and determine $u(x,t)$ for all $t > 0$.

Exercise 2 (Traffic Flow Equation)
Consider the traffic flow equation

$$\frac{\partial q}{\partial t} + \frac{\partial}{\partial x} f(q) = 0, \tag{3}$$

where $q(x,t)$ is the car density, $U(q)$ is the traffic speed as a function of the car density and $f(q) = qU(q)$. A simple model for traffic flow is obtained by taking

$$U(q) = u_m \left(1 - \frac{q}{q_m}\right), \tag{4}$$

with $u_m > 0$ and q_m being the maximum speed and maximum car density, respectively. If the car density is maximal, we say that the traffic is "bumper-to-bumper". At zero density (empty road), the traffic speed is u_m. As q approaches q_m, the speed decreases to zero. The model then reads

$$\frac{\partial q}{\partial t} + \frac{\partial}{\partial x} u_m \left(q - \frac{q^2}{q_m}\right) = 0. \tag{5}$$

We equip (5) with the initial condition

$$q(x,0) = \begin{cases} q_l, & \text{for} \quad x < 0, \\ q_r, & \text{for} \quad x > 0. \end{cases} \tag{6}$$

(a) **The green light problem.** Assume that the traffic is standing at a red light, placed at $x = 0$, while the road ahead is empty. At time $t = 0$, the traffic light turns green and we want to describe the car flow evolution for $t > 0$. To represent this situation, we set $q_l = q_m$ and $q_r = 0$ in (6). Draw the profile of $q_0(x)$ and sketch the characteristics of the entropy solution $q(x,t)$ in the $x-t$ plane, and determine $u(x,t)$ for all $t > 0$.

(b) **Traffic jam ahead.** We now consider the initial density profile with $q_l = \frac{1}{8} q_m$ and $q_r = q_m$. For $x > 0$, the density is maximal and therefore the traffic is "bumper-to-bumper". The cars on the left move with speed $U = \frac{7}{8} u_m$, so that we expect congestion. Draw the profile of $q_0(x)$ and sketch the characteristics of the entropy solution $q(x,t)$ in the $x-t$ plane, and determine $q(x,t)$ for all $t > 0$.

(c) **Entropy Solutions.** Show that for the traffic flow equation (5), the condition $q_l < q_r$ is required for a shock to be admissible. Do this by verifying each of the following conditions:

   (i) The entropy condition $f'(q_l) > s > f'(q_r)$.

(ii) There exists an entropy function $\eta(q)$ and a corresponding entropy flux $\psi(q)$ such that

$$s(\eta(q_r) - \eta(q_l)) \geq \psi(q_r) - \psi(q_l) \tag{7}$$

holds if and only if $q_l < q_r$.

Exercise 3 (Liu's Entropy Condition)
Consider the conservation law

$$\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = 0 \tag{8}$$

where $f(u) = e^u$. We discuss the Riemann problem

$$u_0(x) = \begin{cases} u_l, & \text{for} \quad x < 0, \\ u_r, & \text{for} \quad x > 0 \end{cases} \tag{9}$$

1. Show that when $u(x,t)$ is a weak solution and $\lambda > 0$, then $u(\lambda x, \lambda t)$ is a weak solution too.
2. Suppose that the solution has a shock wave with values u_l and u_r to the left and to the right, respectively. Which values u_l and u_r are admissible on the basis of Liu's entropy condition?

Remark: the scale invariance above suggests that "physical" solutions to the Riemann problem should be scale-invariant themselves.