Exercise Set 2: More on Characteristics and Weak solutions

Exercise 1 (Method of Characteristics I)
Suppose that the flux $f(u, x, t)$ is differentiable in all variables. Find curves along which the conservation law

$$\frac{\partial u(x,t)}{\partial t} - x\frac{\partial f(u(x,t),x,t)}{\partial x} = 0 \tag{1}$$

can be written as a collection of ordinary differential equations.

Exercise 2 (Method of Characteristics II)

(i) Consider the conservation law

$$\frac{\partial u}{\partial t} - x\frac{\partial u}{\partial x} = 0 \tag{2}$$

with initial value

$$u(x,0) = x. \tag{3}$$

Sketch the characteristics up to time $t = 1$. Describe the graph of the function $u(\cdot, t)$ as t increases.

(ii) Consider the conservation law

$$\frac{\partial u}{\partial t} + x\frac{\partial u}{\partial x} = 0 \tag{4}$$

with initial value

$$u(x,0) = x. \tag{5}$$

Draw the characteristics and describe the graph of the function $u(\cdot, t)$ as t increases.

Exercise 3 (Weak Solutions of the Linear Transport Equation)
Show that a weak solution to the linear transport equation

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0,$$

with $a \in \mathbb{R}$ and initial data

$$u(x,0) = \begin{cases} 1, & \text{for} \quad x < 0, \\ 0, & \text{for} \quad x > 0, \end{cases} \tag{6}$$

is given by

$$u(x,t) = \begin{cases} 1, & \text{for} \quad x < at, \\ 0, & \text{for} \quad x > at, \end{cases} \tag{7}$$

Exercise 4 (Rarefaction Waves)
Consider the initial value problem

$$\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = 0, \quad u(x,0) = u_0(x), \tag{8}$$

with $f(u) = \frac{u^2}{2}$, and

$$u_0(x) = \begin{cases} 2, & 0 < x < 1, \\ 0, & \text{otherwise,} \end{cases} \tag{9}$$

Here a rarefaction wave arises at one discontinuity and a shock at the other. The goal of this exercise is to determine the exact solution for all $t > 0$. In this setup, the rarefaction wave catches up with the shock at some time $T_c > 0$.

(i) Draw the profile of $u_0(x)$ and sketch the characteristics in the strip $0 < t < T_c$ of the $x - t$ plane.

(ii) Determine the exact solution for $0 < t < T_c$.

(iii) Let $x_s(t)$ be shock's location at $t > T_c$. By using the Rankine-Hugoniot jump condition construct an ODE to determine $x_s(t)$ for all $t > T_c$. In the sketch you drew in (i), extend the characteristic lines to $t > T_c$.