Characteristics and Weak solutions

Exercise 1
The partial differential equation

$$u_t + f(u)_x = 0 \ , \quad u(x,0) = u_0(x) \tag{1}$$

with $f(u) = u^2/2$ is known as (the inviscid) Burgers equation. Draw the characteristics of the solution in the x-t plane, driven by the initial condition

$$u(x,0) = u_0(x) := \begin{cases} 0 & x < -1 \\ 1+x & -1 < x < 0 \\ 1-x & 0 < x < 1 \\ 0 & 1 < x \end{cases} \ . \tag{2}$$

Compute the exact solution in $0 < t < 1$ and draw its profile at $t = 1$.

Exercise 2
Consider the following IVP for the Burgers equation

$$u_t + \left(\frac{u^2}{2}\right)_x = 0 \quad u(x,0) = u_0(x) = \begin{cases} u_l & x < 0 \\ u_r & 0 < x \end{cases} \ . \tag{3}$$

This is known as a *Riemann problem.* For $u_l < u_r$, and any $u_m \in (u_l, u_r)$, let

$$u(x,t) = \begin{cases} u_l & x < s_m t \\ u_m & s_m t < x < u_m t \\ x/t & u_m t < x < u_r t \\ u_r & u_r t < x \end{cases} \ , \tag{4}$$

where $s_m = (u_m + u_l)/2$. Show that u is a weak solution of (3), and draw its characteristics. Can you give the expression of any other weak solution for this problem?

Exercise 3
In this example the flux f depends on x as well as on the solution u. Suppose u is the solution of the PDE

$$u_t + (a(x)u)_x = 0, \quad a(x) = x \tag{5}$$

which satisfies the initial condition $u(x,0) = u_0(x)$, where u_0 is some given smooth function. Use the method of characteristics to compute the equation describing the characteristics in the x-t plane. Is the solution u constant on the characteristic curves?

Information
A function u is said to be a *weak solution* to the initial value problem (IVP)

$$u_t + f(u)_x = 0 \ , \quad u(x,0) = u_0(x) \tag{6}$$

if for any compactly supported C^1 function $\phi = \phi(x,t)$, there holds

$$\int_0^\infty \int_{-\infty}^\infty \Big(\phi_t u + \phi_x f(u)\Big) \mathrm{d}\, x \mathrm{d}\, t = -\int_{-\infty}^\infty \phi(x,0) u_0(x) \mathrm{d}\, x \ . \tag{7}$$

Exercise 4
Suppose $f(u) = au$, where $a > 0$ is a constant and $u_0(x)$ is an integrable function. Verify that $u(x) = u_0(x - at)$ satisfies (6) in integral form:

$$\int_{x_1}^{x_2} u(x,t_2)\, dx - \int_{x_1}^{x_2} u(x,t_1)\, dx = -\int_{t_1}^{t_2} f(u(x_2,t))\, dt + \int_{t_1}^{t_2} f(u(x_1,t))\, dt \ . \tag{8}$$

References

[1] Partial Differential Equations, by Lawrence. C. Evans. *Chapter 3, Section 2.*

[2] Partial Differential Equations, by Fritz John.