Exercise Set 13: Discontinuous Galerkin

Let $=_x \times_t$, where $_x = (-1, 1)$ is the spacial domain and $_t = (0, T)$ is the time domain.

Exercise 1 1. Consider the scalar problem

$$u_t + au_x = bu \quad x \in (-1, 1) \tag{1}$$

with proper initial conditions and a and b being real constants. **(a)** Propose a discontinuous Galerkin method for solving this problem. **(b)** Prove that the semi-discrete scheme is stable.

2. Consider the scalar PDE

$$u_t + au_x = g(x, t) \tag{2}$$

in , with proper initial conditions and periodic boundary conditions. Here $a > 0$ is a real constant. Write the weak discontinuous Galerkin (DG) formulation for the problem. Pick an appropriate numerical flux and prove that the semi-discrete scheme is stable. (Hint: To prove stability, consider the problem for the error between the computed and the exact solution - known as the error equation).

Consider the system

$$u_t + v_x = 0 \tag{3}$$

$$v_t + u_x = 0 \tag{4}$$

in , subject to periodic boundary conditions. Write the weak DG formulation for the problem. Show that the formulation with the upwind flux is stable.

Consider the ODE system

$$u' = L(u)\ . \tag{5}$$

Suppose there exists a positive constant k_{FE} such that the forward Euler method

$$v^{n+1} = v^n + kL\left(v^n\right)$$

with $0 < k \leq k_{FE}$, applied to (5) satisfies

$$\|v^{n+1}\| \leq \|v^n\| \tag{6}$$

in some norm $\|\cdot\|$. Show that the numerical approximation U of (5) obtained by the following 3rd-order Runga-Kutta method

$$\begin{aligned} U^{(1)} =& U^n + kL\left(U^n\right) \\ U^{(2)} =& \frac{3}{4}U^n + \frac{1}{4}U^{(1)} + \frac{1}{4}kL\left(U^{(1)}\right) \\ U^{n+1} =& \frac{1}{3}U^n + \frac{2}{3}U^{(2)} + \frac{2}{3}kL\left(U^{(2)}\right) \end{aligned}$$

satisfies (6), provided $0 < k \leq k_{FE}$.