Exercise 12: WENO reconstruction

Exercise 1
The WENO reconstruction is based on a convex combination, with coefficients ω_r, of approximations $v^{(r)}_{i+1/2}$ calculated on k different stencils. The coefficients ω_r depend on another set of coefficients d_r and some smoothness indicators β_r.

1. The Smoothness indicators β_r are defined by

$$\beta_r = \sum_{l=1}^{k-1} \Delta x^{2l-1} \int_{x_i-1/2}^{x_{i+1/2}} \left(\frac{\mathrm{d}^{\,l} p_r}{\mathrm{d}\, x^l}(x) \right)^2 \mathrm{d}\, x \qquad r = 0, \ldots, k-1 \ . \tag{1}$$

   For $k = 2$ (3rd-order reconstruction), this yields the following expressions:

$$\beta_0 = \left(\overline{U}_{i+1} - \overline{U}_i \right)^2 \ , \qquad \beta_1 = \left(\overline{U}_i - \overline{U}_{i-1} \right)^2 \ . \tag{2}$$

   Verify (2).

2. The coefficients d_r are chosen so that

$$\sum_{r=0}^{k-1} d_r v^{(r)}_{i+1/2} = v\left(x_{i+1/2}\right) + O\left(\Delta x^{2k-1}\right) \ . \tag{3}$$

   For $k = 2, 3$,

$$k = 2: \quad d_0 = \frac{2}{3} \ , \quad d_1 = \frac{1}{3} \tag{4}$$

$$k = 3: \quad d_0 = \frac{3}{10} \ , \quad d_1 = \frac{3}{5} \ , \quad d_2 = \frac{1}{10} \ . \tag{5}$$

   Explain how these values are obtained and verify (4).

3. Write a program that implements WENO reconstruction for both $k = 2, 3$.

4. Write a finite volume code for

$$u_t + a u_x = 0$$

   where the interface values are obtained by using the WENO reconstruction. Plug these values in the Godunov flux, and integrate the semi-discrete scheme using SSP-RK3. Implement the following initial conditions on the domain $[-1, 1]$ with periodic BC

$$v_0(x) = \sin(\pi x) \ , \quad T_f = 5 \tag{6}$$

$$v_0(x) = \begin{cases} 1 & |x| < 0.5 \\ -1 & |x| > 0.5 \end{cases} \ , \quad T_f = 0.5. \tag{7}$$

   Use a CFL of 0.5 to evaluate the time-step.

5. Run the code for $k = 2, 3$ and $a = 1$. Do you recover the expected order of convergence for (6)? Is the solution oscillatory for (7)?