Exercise Set 11: ENO reconstruction

Exercise 1
Consider the linear advection equation

$$v_t + av_x = 0 \; . \tag{1}$$

In previouses exercises, we had observed that a fixed stencil approach for reconstruction is not suitable for obtaining stable non-oscillatory solutions. In the following exercises you are asked to write a number of programs which calculate different parts of the ENO reconstruction. Assume the grid is uniform.

1. Write a program that takes as an input a stencil-size k, and a grid function v, and finds for each cell an appropriate stencil of size k. For example, for the ith cell $I_i = (x_{i-1/2}, x_{i+1/2})$, the function should return the output r_i, if the chosen stencil is $x_{i-r_i}, \dots, x_{i-r_i+k-1}$. Here, the value v_i of v at x_i is interpreted as average of the approximated function v in I_i. Notice that since the grid is assumed to be uniform, you can make your program slightly more efficient by computing the *undivided differences*,

$$V\big\langle x_{i-1/2}, x_{i+1/2}\big\rangle := V\big(x_{i-1/2}, x_{i+1/2}\big) = v_i \tag{2}$$

$$V\big\langle x_{i-1/2}, \dots, x_{i+j+1/2}\big\rangle := V\big\langle x_{i+1/2}, \dots, x_{i+j+1/2}\big\rangle - V\big\langle x_{i-1/2}, \dots, x_{i+j-1/2}\big\rangle \tag{3}$$

instead of the divided differences.

2. Write a program that calculates the coefficients

$$c_{rj} = \sum_{m=j+1}^{k} \frac{1}{\prod_{\substack{l=0\\ l\neq m}}^{k} (m-l)} \sum_{\substack{l=0\\ l\neq m}}^{k} \prod_{\substack{q=0\\ q\neq m,l}}^{k} (r-q+1) \qquad j = 0, \dots, k-1, \qquad r = -1, 0, \dots, k-1 \; . \tag{4}$$

Explain how this expression is derived. Check that this formula generates the coefficients obtained in Exercise 10 for $k = 3$.

3. Write a program that implements the ENO reconstruction. The program should take a grid function v, and a stencil-size k, and return the values

$$v^-_{i+1/2} = \sum_{j=0}^{k-1} c_{r_i,j} v^n_{i-r_i+j} \; , \qquad\qquad v^+_{i-1/2} = \sum_{j=0}^{k-1} c_{r_i-1,j} v^n_{i-r_i+j} \; . \tag{5}$$

You may assume the function is periodic.

4. Write a finite volume code for (1), where the interface values can be obtained by using the ENO reconstruction. Plug these values in the Godunov flux, and integrate the semi-discrete scheme using SSP-RK3. Implement the following initial conditions on the domain $[-1, 1]$ with periodic BC

$$v_0(x) = \sin(\pi x) \; , \quad T_f = 5 \tag{6}$$

$$v_0(x) = \begin{cases} 1 & |x| < 0.5 \\ -1 & |x| > 0.5 \end{cases} \; , \quad T_f = 0.5. \tag{7}$$

Use a CFL of 0.5 to evaluate the time-step.

5. Run the code for $k = 3$ and $a = 1$. Do you recover 3rd-order convergence for (6)? Is the solution oscillatory for (7)?