Numerical Analysis and Computational Mathematics

Fall Semester 2024 – CSE Section

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Session 9 – November 13, 2024

Linear systems: direct and iterative methods

Exercise I (MATLAB)

Consider the linear system $A\mathbf{x} = \mathbf{b}$ with $A \in \mathbb{R}^{n\times n}$, $\mathbf{x}$, $\mathbf{b} \in \mathbb{R}^n$ for $n \geq 1$. We are interested in computing the solution $\mathbf{x}$ by means of the LU factorization method.

a) Consider the two matrices

$$A_1 = \begin{bmatrix} 3 & -2 & 0 & & & & \\ -1 & 3 & -2 & 0 & & & \\ 0 & -1 & 3 & -2 & 0 & & \\ & & \ddots & \ddots & \ddots & & \\ & & & & & & \\ & & & 0 & -1 & 3 & -2 \\ & & & & 0 & -1 & 3 \end{bmatrix}, \qquad A_2 \ : \ (A_2)_{ij} = \frac{1}{i+j-1}, \quad i,j = 1,\ldots n.$$

The matrix A_2 is the *Hilbert matrix* of dimension n. Generate the matrices A_1 and A_2 for $n = 4$ in MATLAB (for A_2 you can use the MATLAB command `hilb`), and determine if there exists a unique LU factorization of the matrices without performing the pivoting technique. (*Hint*: you may use the MATLAB function `eig` to compute eigenvalues.)

b) Set $n = 9$ and compute by means of the LU factorization method the solutions $\mathbf{x}_1$ and $\mathbf{x}_2$ of the linear systems $A_1\mathbf{x}_1 = \mathbf{b}_1$ and $A_2\mathbf{x}_2 = \mathbf{b}_2$, where $\mathbf{x}_1 = \mathbf{x}_2 = (1, 1, \ldots, 1)^T$, respectively. (*Hint*: see Series 8.)

Compute the relative errors $e_{\text{rel},i} := \frac{\|\mathbf{x}_i - \widehat{\mathbf{x}}_i\|}{\|\mathbf{x}_i\|}$, where $\widehat{\mathbf{x}}_i$ are the approximate solutions of the linear systems. Similarly, compute the relative residuals $r_{\text{rel},i} := \frac{\|\mathbf{r}_i\|}{\|\mathbf{b}_i\|}$, where $\mathbf{r}_i := \mathbf{b}_i - A_i\widehat{\mathbf{x}}_i$, and the condition numbers $K_2(A_i)$, for $i = 1, 2$. Motivate the results obtained.

c) Repeat point b) for $n = 4, 5, \ldots, 13$. Plot the relative errors, the relative residuals, and the condition numbers vs. n and comment the results obtained. Can we consider the approximate solutions $\widehat{\mathbf{x}}_i$ satisfactory for $n = 13$?

Exercise II (MATLAB)

Consider the Jacobi and Gauss-Seidel iterative methods for the solution of linear system $A\mathbf{x} = \mathbf{b}$ with $A \in \mathbb{R}^{n\times n}$ and $\mathbf{x}$, $\mathbf{b} \in \mathbb{R}^n$.

a) Consider the non-singular matrices:

$$A_1 = \begin{bmatrix} 3 & -2 & 1 \\ 2 & 1.65 & -1 \\ 0 & 1 & 4 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} 5 & -3 & -2 \\ -3 & 3 & 0 \\ -2 & 0 & 4 \end{bmatrix},$$

$$A_3 = \begin{bmatrix} 4 & -1 & 0 & & & & \\ -1 & 4 & -1 & 0 & & & \\ 0 & -1 & 4 & -1 & 0 & & \\ & & \ddots & \ddots & \ddots & & \\ & & & 0 & -1 & 4 & -1 \\ & & & & 0 & -1 & 4 \end{bmatrix} \in \mathbb{R}^{n_3\times n_3}, \quad \text{with } n_3 = 100.$$

Note that the diagonal elements of the previous matrices are non-zero. Are the Jacobi and Gauss-Seidel methods convergent for any choice of the initial solution $\mathbf{x}^{(0)}$ for the matrices A_1, A_2, and A_3? Motivate the answer. (*Hint*: in some cases, it may be sufficient to inspect the matrices A_p without assembling the iteration matrices. Otherwise, you should compute the spectral radii of the iteration matrices. To extract a lower or upper triangular matrix from a given matrix, you may use the MATLAB functions `tril` and `triu`, respectively.)

b) Write the MATLAB functions `jacobi.m` and `gauss_seidel.m`, which implement the Jacobi and Gauss-Seidel methods for solving the generic linear system $A\mathbf{x} = \mathbf{b}$. Implement a stopping criterion based on the norm of the residual: $r^{(k)} = \|\mathbf{r}^{(k)}\| = \left\|A\mathbf{x}^{(k)} - \mathbf{b}\right\| < tol$. You can use the functions `jacobi_template.m` and `gauss_seidel_template.m` as templates:

```
function [ x, k, res ] = jacobi( A, b, x0, tol, kmax )
% JACOBI solve the linear system A x = b by means of the
% Jacobi iterative method; diagonal elements of A must be nonzero.
% Stopping criterion based on the residual.
%  [ x, k, res ] = jacobi( A, b, x0, tol, kmax )
%  Inputs:  A    = matrix (square matrix)
%           b    = vector (right hand side of the linear system)
%           x0   = initial solution (colum vector)
%           tol  = tolerence for the stopping driterion based on residual
%           kmax = maximum number of iterations
%  Outputs: x    = solution vector (column vector)
%           k    = number of iterations at convergence
%           res  = value of the norm of the residual at convergence
%

return
```

```
function [ x, k, res ] = gauss_seidel( A, b, x0, tol, kmax )
% GAUSS_SEIDEL solve the linear system A x = b by means
% of the Gauss-Seidel iterative method; diagonal elements of A
```

```
% must be nonzero. Stopping criterion based on the residual.
%  [ x, k, res ] = gauss_seidel( A, b, x0, tol, kmax )
%  Inputs:  A    = matrix (square matrix)
%           b    = vector (right hand side of the linear system)
%           x0   = initial solution (colum vector)
%           tol  = tolerence for the stopping driterion based on residual
%           kmax = maximum number of iterations
%  Outputs: x    = solution vector (column vector)
%           k    = number of iterations at convergence
%           res  = value of the norm of the residual at convergence
%

return
```

c) If possible, solve the linear systems $A_p \mathbf{x}_p = \mathbf{b}_p$, for $p = 1, 2, 3$, using the Jacobi and Gauss-Seidel methods. In all cases, define the right-hand-sides $\mathbf{b}_p$ so that the exact solution is $\mathbf{x}_p = (1, 1, \dots, 1)^T \in \mathbb{R}^{n_p}$. Use $tol = 10^{-6}$, `kmax` $= 1000$, and $\mathbf{x}_p^{(0)} = (0, 0, \dots, 0)^T \in \mathbb{R}^{n_p}$. Report the norm of the errors $e_p^{(k_p)} = \left\| \mathbf{e}_p^{(k_p)} \right\| = \left\| \mathbf{x}_p - \mathbf{x}_p^{(k_p)} \right\|$, the number of iterations k_p necessary for convergence, and the norm of the residual $r_p^{(k_p)} = \left\| \mathbf{r}_p^{(k_p)} \right\|$. Discuss the results in relation to point a).

d) Consider the family of matrices depending on a parameter $\gamma \in \mathbb{R}$, defined as

$$A_4 = \begin{bmatrix} 8 & \gamma & -2 & -1 \\ -2 & 2 & -\gamma & -3 \\ -1 & -2 & 18 & -18 \\ -1 & -3 & -7 & 25 \end{bmatrix}, \qquad \text{with } \gamma \in [-10, 25].$$

Note that the matrix A_4 is non-singular for the given range of γ. Investigate graphically the (speed of) convergence of the Jacobi and Gauss-Seidel methods, when applied to solve linear systems with A_4 as matrix, as γ varies (take small steps of γ from -10 to 25). Which of the two methods would you choose for $\gamma = 0$, 9, and 15? Motivate the answer.