
Numerical Analysis and Computational Mathematics

Fall Semester 2024 – CSE Section

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Nonlinear equations: Newton method

Exercise I (MATLAB)

With the help of the file `newton_template.m`, implement the Newton method in a MATLAB function, to be saved as `newton.m`. The layout of the function is the following:

```
function [xvect, resvect, nit] = newton( fun, dfun, x0, tol, nmax )
% NEWTON Find a zero of a nonlinear scalar function.
% [XVECT] = NEWTON(FUN,DFUN,X0,TOL,NMAX) finds a zero of the differentiable
% function FUN using the Newton method and returns a vector XVECT containing
% the successive approximations of the zero (iterates). DFUN is the derivative of FUN.
% FUN and DFUN accept real scalar input x and return a real scalar value;
% FUN and DFUN can also be inline objects. X0 is the initial guess.
% TOL is the tolerance on error allowed and NMAX the maximum number of iterations.
% The stopping criterion based on the difference of successive iterates is used.
% If the search fails a warning message is displayed.
%
% [XVECT,RESVECT,NIT] = NEWTON(FUN,DFUN,X0,TOL,NMAX) also returns the vector
% RESVECT of residual evaluations for each iterate, and NIT the number of iterations.
% Note: the length of the vectors is equal to ( NIT + 1 ).
%
return
```

As a stopping criterion for the Newton method, check if the difference of successive approximate solutions at step n is smaller than the prescribed tolerance tol , i.e. $|x^{(n)} - x^{(n-1)}| < tol$, with a limit on the maximum number of iterations n_{max} ($n \leq n_{max}$).

- Use the function `newton` to find the zero $\alpha = 0$ of the nonlinear function $f(x) = \sin(2x) + x$ in $(-1, 1)$ starting from $x^{(0)} = 0.7$, with the tolerance $tol = 10^{-5}$ and $n_{max} = 50$. How many iterations (n_c) are required for the convergence of the Newton method, and how large is the error $e^{(n_c)} = |x^{(n_c)} - \alpha|$?
- What is the expected convergence order of the Newton method to the zero α for the function $f(x)$? Motivate the answer based on the theoretical convergence results.

- c) Set $x^{(0)} = 0.7$, $n_{max} = 6$ and $tol = 10^{-12}$. Plot in semi-logarithmic scale the errors $e^{(n)} = |x^{(n)} - \alpha|$ vs. the iteration number n . By comparing the result with the plot of $b_n = 2^{-n}$, what can we deduce about the convergence order of the Newton method applied to the function $f(x)$?
- d) Modify the function `newton` to implement the stopping criterion based on the residual, i.e. $|r^{(n)}| = |f(x^{(n)})| < tol$ and save it in a different file `newton_residual.m`. Apply it to find the unique zero $\alpha = 0$ of the function $f(x) = \exp(\beta x) - 1$, with $\beta \in \{10^{-3}, 1, 10^3\}$. Set the initial value $x^{(0)} = 0.1$, the tolerance $tol = 10^{-7}$, and $n_{max} = 150$. Compare the values of the (absolute) residual $|r^{(n_c)}|$ and error $e^{(n_c)}$ at convergence for the different values of β . Is the stopping criterion based on the residual satisfactory for these values of β ? Why?

Exercise II (MATLAB)

Consider the Newton method for finding the zero $\alpha = 0$ of the function $f(x) = (\sin(x))^m$ in the interval $(-\pi/2, \pi/2)$, starting from the initial value $x^{(0)} = \pi/6$, for different $m = 1, 2, 3, \dots$

- a) What are the expected convergence orders of the Newton method to the zero α for the function $f(x)$ with $m = 1, 2, 3$? Motivate the answer based on the theoretical convergence results.
- b) Use the function `newton` to find the zero α for $m = 1, 2, 3$. Set $tol = 10^{-8}$ and use the stopping criterion based on the difference of successive approximate solutions and maximum number of iterations $n_{max} = 50$. How many iterations are required to converge to the prescribed tolerance?
- c) Similarly to Exercise I point c), plot the errors obtained for $m = 1, 2, 3$ vs. the iteration number n (set $n_{max} = 50$). Justify the results using point a).
- d) Assume that the function $f(x)$ has a zero α of *known* multiplicity $m > 1$, i.e., given $f(x) \in C^m(I_\alpha)$ (with I_α a neighborhood of α),

$$f(\alpha) = f'(\alpha) = \dots = f^{(m-1)}(\alpha) = 0 \quad \text{and} \quad f^{(m)}(\alpha) \neq 0.$$

How should the Newton method be modified to account for this? Implement the modified Newton method in a function `newton_modified`, taking as additional input the multiplicity m of the zero α .

- e) What are the expected convergence orders of the modified Newton method applied to the function $f(x)$ in the cases $m = 1, 2, 3$? Repeat point c) by using the function `newton_modified` and comment on the results obtained.

Exercise III (MATLAB)

Consider the cubic polynomial $p(x) = -3x^3/8 + 5x^2/4 + x/2 - 1$.

- a) Plot the polynomial $p(x)$ on the interval $x \in (-1, 3)$. Plot in the same figure the tangent lines to the curve $(x, p(x))$ at $x = 0$ and $x = 2$.
- b) Use the Newton method to find the zero $\alpha \in (0, 2)$, starting from the initial value $x^{(0)} = 10^{-3}$. Set $tol = 10^{-8}$ and $n_{max} = 20$. Does the method converge? To which value? In how many iterations?

- c) Repeat point b) by setting $x^{(0)} = -10^{-3}$ and $x^{(0)} = 0$. What do you observe? Motivate the results with the help of the plot from point a).

Exercise IV (MATLAB)

With the help of the file `newtonsys.template.m`, complete the implementation of the Newton method for systems of nonlinear equations in a MATLAB function `newtonsys` (save the file as `newtonsys.m`). The layout of the function is the following:

```
function [x, res, nit] = newtonsys( F, J, x0, tol, nmax )
% NEWTONSYS Find the zeros of a system of nonlinear equations.
% [X] = NEWTONSYS(F,J,X0,TOL,NMAX) find the zero X of the
% continuous and differentiable system of functions F nearest to X0 using the
% Newton method. J is a function which takes X and returns the Jacobian matrix.
% X0 is a column vector; F returns a column vector and J a square matrix.
% The stopping criterion is based on the difference (norm) of successive
% iterates.
% If the search fails a warning message is displayed.
%
% [X,RES,NITER] = NEWTONSYS(F,J,X0,TOL,NMAX) returns the value of the
% residual RES in X and the number of iterations NITER required for computing X.
% Note: only the final iterate is stored in X; similarly for RES.
%
return
```

As stopping criterion for the Newton method, consider the test on the increment of successive approximations $\mathbf{x}^{(n)}$ at the iteration step n , i.e. $\|\mathbf{x}^{(n)} - \mathbf{x}^{(n-1)}\|_2 < tol$ for a prescribed tolerance tol , with a limit on the maximum number of iterations n_{max} ($n < n_{max}$). *Hint:* use the MATLAB command `norm` to compute the norm of a vector, and `\` to solve systems of linear equations (see `help mldivide`).

Use the function `newtonsys` to find the zero $\boldsymbol{\alpha} \in \mathbb{R}^d$ of the system of nonlinear equations $\mathbf{F}(\mathbf{x}) = \mathbf{0}$, with $\mathbf{x} = (x_1, \dots, x_d)^T \in \mathbb{R}^d$, $\mathbf{F} : \mathbb{R}^d \rightarrow \mathbb{R}^d$, where $d = 2$ and

$$\mathbf{F}(\mathbf{x}) = \begin{bmatrix} e^{x_1^2 + x_2^2} - 1 \\ e^{x_1^2 - x_2^2} - 1 \end{bmatrix}.$$

(*Hint:* you may define multivariate anonymous functions by using standard array indexing, e.g. `f=@(x) x(1)+x(2); .` Also, you may define vector- and matrix-valued anonymous functions by using square bracket notation, e.g. `F=@(x) [x; 2*x];` and `J=@(x) [x 2*x; 1 -x]; .`)

Find the zero $\boldsymbol{\alpha} = (0, 0)^T$ by setting $tol = 10^{-5}$ and $n_{max} = 100$, from the two initial points $\mathbf{x}^{(0)} = (1.5, -2)^T$ and $\mathbf{x}^{(0)} = (4, 4)^T$. Report the number of iterations required for the convergence of the method and discuss the choice of $\mathbf{x}^{(0)}$.