Numerical Analysis and Computational Mathematics

Fall Semester 2024 – CSE Section

Prof. Laura Grigori

Assistant: Israa Fakih

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Ordinary differential equations

Exercise I (MATLAB)

Consider the system of first order ODEs

$$\text{find } \mathbf{y} \; : I \subset \mathbb{R} \to \mathbb{R}^m \qquad : \qquad \begin{cases} \mathbf{y}'(t) = \mathbf{F}(t, \mathbf{y}(t)) & \text{for all } t \in I, \\ \mathbf{y}(t_0) = \mathbf{y}_0, \end{cases} \tag{1}$$

where $m \geq 1$, $I = (t_0, t_f)$ is the integration interval, $\mathbf{F} \; : I \times \mathbb{R}^m \to \mathbb{R}^m$ is a given vector-valued function, and $\mathbf{y}_0 \in \mathbb{R}^m$ is the initial datum. To define a Lotka-Volterra prey-predator model for the dynamics of populations, we set $m = 2$, $\mathbf{y}(t) = (y_1(t), y_2(t))^T$, and choose:

$$\mathbf{F}(t, \mathbf{y}) = \left[\begin{array}{c} C_1 \, y_1 \, (1 - b_1 \, y_1 - d_2 \, y_2) \\ -C_2 \, y_2 \, (1 - b_2 \, y_2 - d_1 \, y_1) \end{array} \right].$$

In particular, we select $C_1 = 0.15$, $C_2 = 0.075$, $b_1 = 0.002$, $b_2 = 0$, $d_1 = 0.0210$, $d_2 = 0.0325$, $t_0 = 0$, $t_f = 600$, and $\mathbf{y}_0 = (55, 20)^T$. The solution $\mathbf{y}(t)$ tends to the equilibrium state $\mathbf{y}_E \simeq (47.62, 27.84)^T$ for $t \to \infty$.

a) Solve the prey-predator model with the specified data by means of the forward Euler method implemented in the function `forward_euler_system.m` from Series 13. Set $N_h = 5000$ and plot the numerical solution vs t.

b) Plot the trajectory of the solution in the phase space (prey-predator).

c) Repeat points a) and b) for $y_0 = (35, 40)^T$.

Exercise II (MATLAB)

Consider the 2-dimensional truss bridge introduced in Series 11, and displayed in Fig. 1. This time, our goal consists in simulating the dynamics of the bridge under the action of external forces.
At any time t, we denote by $\mathbf{u}_i(t) \in \mathbb{R}^2$ the displacement vector of each node $i = 1, \ldots, N_{nodes}$ and by $\mathbf{f}_i^{ext}(t) \in \mathbb{R}^2$ the external forces acting at the nodes. Elastic internal forces $\mathbf{f}_{i,j}^{int}(t) \in \mathbb{R}^2$ and damping forces $\mathbf{f}_i^{c}(t) \in \mathbb{R}^2$ are also included in the model.

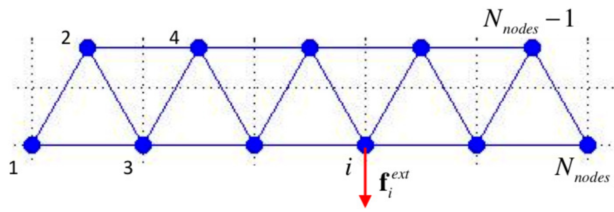

Figure 1: Truss bridge model.

At each node i, whose mass is m_i, Newton's second law can be expressed as: $m_i\mathbf{a}(t) + \mathbf{f}_i^c(t) + \sum_{j\in I_i} \mathbf{f}_{i,j}^{int}(t) = \mathbf{f}_i^{ext}(t)$, with $I_i := \{i-2, i-1, i+1, i+2\} \cap \{k\}_{k=1}^{N_{nodes}}$ and $\mathbf{a}_i(t) = \mathbf{u}_i''(t) \in \mathbb{R}^2$ is the acceleration of the node. This can be cast as a second order system of ODEs:

$$\text{find } \mathbf{d}(t) \,:\, (t_0, t_f) \subset \mathbb{R} \to \mathbb{R}^m \quad : \quad \begin{cases} M\,\mathbf{d}''(t) + C\,\mathbf{d}'(t) + K\,\mathbf{d}(t) = \mathbf{b}(t) & \forall t \in (t_0, t_f), \\ \mathbf{d}'(t_0) = \mathbf{v}_0, \\ \mathbf{d}(t_0) = \mathbf{d}_0. \end{cases} \tag{2}$$

Above,

$$\mathbf{d} = ((\mathbf{u}_1(t))^T, (\mathbf{u}_2(t))^T, \ldots, (\mathbf{u}_{N_{nodes}}(t))^T)^T \in \mathbb{R}^{2N_{nodes}}$$

is the unknown solution vector representing the displacements of each node of the bridge. The matrix $K \in \mathbb{R}^{2N_{nodes}\times 2N_{nodes}}$ is the "stiffness matrix" and depends on the elastic properties of the beams by means of the scalar parameter k_{beam}. The matrix K is sparse, symmetric, and can be defined via the MATLAB function `bridge_stiffness_matrix.m` from Series 11. The matrix $M \in \mathbb{R}^{2N_{nodes}\times 2N_{nodes}}$ is the mass matrix, which, in this ("lumped") case, is just a multiple of the identity matrix: $M = m_{node}\, I$. The matrix $C \in \mathbb{R}^{2N_{nodes}\times 2N_{nodes}}$ models the damping forces and is obtained as $C = \alpha\, M + \beta\, K$ for some scalar parameters α and β.
The vector

$$\mathbf{b}(t) = ((\mathbf{f}_1^{ext}(t))^T, (\mathbf{f}_2^{ext}(t))^T, \ldots, (\mathbf{f}_{N_{nodes}}^{ext}(t))^T)^T \in \mathbb{R}^{2N_{nodes}}$$

contains the external forces $\mathbf{f}_i^{ext}(t) \in \mathbb{R}^2$ acting on each node. $\mathbf{d}_0$ and $\mathbf{v}_0 \in \mathbb{R}^m$ represent the initial displacement and velocity of the nodes at time t_0, respectively.

a) Set $N_{nodes} = 29$, $k_{beam} = 10^3$, $m_{node} = 2$, and $\alpha = \beta = 0.01$. Define the sparse matrices M, C, and K.

b) In order to make the structural problem well-posed, we constrain to zero three entries of the vector $\mathbf{d}(t)$, for all times t. As in Series 11, we fix both displacement components of the leftmost node and the vertical displacement of the rightmost node. This will turn the system (2) into one of reduced size $\widetilde{m} = 2N_{nodes} - 3$:

$$\text{find } \widetilde{\mathbf{d}}(t) \,:\, (t_0, t_f) \subset \mathbb{R} \to \mathbb{R}^{\widetilde{m}} \quad : \quad \begin{cases} \widetilde{M}\,\widetilde{\mathbf{d}}''(t) + \widetilde{C}\,\widetilde{\mathbf{d}}'(t) + \widetilde{K}\,\widetilde{\mathbf{d}}(t) = \widetilde{\mathbf{b}}(t) & \forall t \in (t_0, t_f), \\ \widetilde{\mathbf{d}}'(t_0) = \widetilde{\mathbf{v}}_0, \\ \widetilde{\mathbf{d}}(t_0) = \widetilde{\mathbf{d}}_0. \end{cases} \tag{3}$$

c) Rewrite (3) as a first order system of ODEs in non-homogeneous form:

$$\text{find } \widetilde{\mathbf{y}}(t) \,:\, (t_0, t_f) \subset \mathbb{R} \to \mathbb{R}^{2\widetilde{m}} \quad : \quad \begin{cases} \widetilde{\mathbf{y}}'(t) = \widetilde{A}\,\widetilde{\mathbf{y}}(t) + \widetilde{\mathbf{g}}(t) & \forall t \in (t_0, t_f), \\ \widetilde{\mathbf{y}}(t_0) = \widetilde{\mathbf{y}}_0. \end{cases} \tag{4}$$

d) Set $t_0 = 0$, $t_f = 250$, $t_{ref} = 25$, and $\mathbf{d}_0 = \mathbf{v}_0 = \mathbf{0}$. Consider a single vertical external force $\mathbf{f}_{15}^{ext} = (0, -q_k(t))^T$ at the 15^{th} node. We consider 5 different possible options (indexed by k) for the force magnitude q_k:

- $q_1(t) = \begin{cases} t/t_{ref} & t \leq t_{ref}, \\ 1 & t > t_{ref}, \end{cases}$;
- $q_2(t) = \begin{cases} t/t_{ref} & t \leq t_{ref}, \\ 0 & t > t_{ref}, \end{cases}$;
- $q_k(t) = \sin(\omega_k\, t)$, with $\omega_3 = 0.25$, $\omega_4 = 0.4688$, and $\omega_5 = 0.65$.

Solve (4) using the backward Euler method via the function `backward_euler_system_nhcc.m` from Series 13. Set $N_h = 2500$. Plot the numerical approximation of the components of the displacement for the 15^{th} node vs t. Then, use the function `plot_bridge.m` to visualize the deformation of the bridge in time.