

Project for the course “Numerical Integration for Stochastic Differential Equations”

Interacting noisy particle systems

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Academic Year 2024/2025

Let us consider the stochastic basis $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t \geq 0})$, where the filtration $(\mathcal{F}_t)_{t \geq 0}$ satisfies the usual conditions. Denote with $(W^i)_{i=1}^n$ a series of n -independent $\mathbb{R}^l$ -dimensional Brownian motions adapted to the aforementioned filtration. Moreover, we consider $(\xi^i)_{i=1}^n$ a collection of $\mathbb{R}^d$ -valued i.i.d. random variables, which are $\mathcal{F}_0$ -measurable. Furthermore, we denote $\mathcal{P}_p(\mathbb{R}^d)$ for $p > 0$ the set of probability measures μ on $\mathbb{R}^d$ such that

$$\int_{\mathbb{R}^d} |w|^p \mu(dw) < +\infty,$$

where $|\cdot|$ denotes the Euclidean norm in $\mathbb{R}^d$. The goal of this project is to analyze theoretically and simulate numerically a system of n interacting particles $(X_t^{n,1}, \dots, X_t^{n,n})$, driven by stochastic differential equations (SDEs) of the form

$$\begin{aligned} dX_t^{n,i} &= a(X_t^{n,i}, \mu_t^n) dt + b(X_t^{n,i}, \mu_t^n) dW_t^i, \quad X_0^{n,i} = \xi^i, \quad i = 1, 2, \dots, n \\ \mu_t^n &= \frac{1}{n} \sum_{k=1}^n \delta_{X_t^{n,k}}, \end{aligned} \tag{1}$$

where $a: \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \rightarrow \mathbb{R}^d$, $b: \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \rightarrow \mathbb{R}^{d \times l}$ are the drift and the diffusion coefficients, respectively, whereas μ_t^n is the so-called empirical measure. Equation (1) represents an *interacting noisy particle systems*, i.e. a system of n particles $(X_t^{n,i})_i$ affected by noise, where the evolution of each particle i is driven by the same drift and diffusion terms, and is influenced by the position of all other particles through the empirical measure μ_t^n . This is an example of law-dependent process and this property carries additional difficulties in the numerical simulation of SDEs like (1).

- **(Q1)** Let us consider the following particle system (called Consensus Based Dynamics - CBD)

$$\begin{aligned} dX_t^{n,i} &= -q(X_t^{n,i} - \bar{X}_t^n) dt + dW_t^i, \quad X_0^{n,i} = \xi^i, \quad i = 1, 2, \dots, n \\ \bar{X}_t^n &= \frac{1}{n} \sum_{k=1}^n X_t^{n,k} \end{aligned}$$

for $q > 0$. The CBD is an example of interacting noisy particle systems. For this kind of dynamics, it is often very interesting to consider the case when the number of particles is very large, i.e. $n \rightarrow +\infty$.

1. Prove that $\lim_n \bar{X}_t^n = \mathbb{E}[\xi^1]$ for all t and almost surely (a.s.).

Hint: write an SDE for $\bar{X}_t^n$.

2. Prove that $\lim_n X_t^{n,i} = Y^i$ a.s., where

$$dY_t^i = -q(Y_t^i - \mathbb{E}[\xi^i]) dt + dW_t^i, \quad Y_0^i = \xi^i. \tag{2}$$

3. Show that $\mathbb{E}[Y_t^i] = \mathbb{E}[\xi^i]$ for all t .

- **(Q2)** Equation (2) is an example of a McKean-Vlasov SDE (MV-SDE), i.e. a dynamics whose coefficients also depend on the law of the solution. A general McKean-Vlasov SDE has the following form:

$$\begin{aligned} dY_t &= a(Y_t, \mu_t^Y) dt + b(Y_t, \mu_t^Y) dW_t, \quad Y_0 = \xi \\ \mu_t^Y &= \mathcal{L}(Y_t), \quad \forall t \geq 0 \end{aligned} \tag{3}$$

where $\mu^Y = (\mu_t^Y)_{t \in [0, T]}$ denotes the law of Y , i.e. $\mathcal{L}(Y)$. We say that the pair (Y, μ) is a *strong solution* of (3) if Y is a continuous adapted process, μ is a probability measure on $C([0, T]; \mathbb{R}^d)$, such that (3) is a.s. satisfied.

Moreover, given $\mathcal{P}_p(\mathcal{X})$, with $(\mathcal{X}, d)$ a metric space and $p > 0$, we define the p -Wasserstein distance between two measures $\mu, \eta \in \mathcal{P}_p(\mathcal{X})$ as

$$\mathcal{W}_p(\mu, \eta) = \left(\inf_{\pi \in \Pi(\mu, \eta)} \int_{\mathcal{X} \times \mathcal{X}} d(x, y)^p \pi(dx, dy) \right)^{\frac{1}{p}},$$

where Π is the set of all the couplings between μ and η , i.e. all joint measures $\pi \in \mathcal{P}_p(\mathcal{X} \times \mathcal{X})$ s.t. $\pi(A \times \mathcal{X}) = \mu(A)$ and $\pi(\mathcal{X} \times A) = \eta(A) \forall A \in \mathcal{B}(\mathcal{X})$. If $(\mathcal{X}, d)$ is complete, then $(\mathcal{P}_p(\mathcal{X}), \mathcal{W}_p)$ is a complete and separable metric space.

Assumption 1. *We consider the following assumptions to be fulfilled:*

- the initial states ξ and $(\xi^i)_{i \in \mathbb{N}}$ are i.i.d. with finite second moment. Moreover, the Brownian motions $(W^i)_{i \in \mathbb{N}}$ are independent and m -dimensional.
- the drift $a : \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \rightarrow \mathbb{R}^d$ and $b : \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \rightarrow \mathbb{R}^{d \times l}$ are Lipschitz, in the sense that there exists a constant $L > 0$ such that

$$|a(x, z) - a(x', z')| + |b(x, z) - b(x', z')| \leq L(|x - x'| + \mathcal{W}_2(z, z')), \quad \forall x, x' \in \mathbb{R}^d, \forall z, z' \in \mathcal{P}_2(\mathbb{R}^d)$$

- the drift a and the diffusion b satisfy the following linear-growth bound condition

$$|a(x, z)|^2 + \|b(x, z)\|_{\mathbb{F}}^2 \leq C_{\text{lgb}}(1 + |x|^2 + \int_{\mathbb{R}^d} |y|^2 dz(y)), \quad \forall x \in \mathbb{R}^d, \forall z \in \mathcal{P}_2(\mathbb{R}^d)$$

for some constant $C_{\text{lgb}} > 0$.

Prove the existence and uniqueness of strong solutions of (3) under Assumption 1.

Hints: the prove of well-posedness of (3) is very similar to the one of standard SDEs (see [4, Theorem 4.5.3] or [5, Section 2.3]). You can follow these steps:

1. Define the map $F : \mathcal{P}_2(C([0, T]; \mathbb{R}^d)) \rightarrow \mathcal{P}_2(C([0, T]; \mathbb{R}^d))$, where $C([0, T]; \mathbb{R}^d)$ is endowed with the uniform convergence metric, as $F(\mu) := \mathcal{L}(Y^\mu)$, with Y^μ solution of the following SDE

$$dY_t^\mu = a(Y_t^\mu, \mu_t) dt + b(Y_t^\mu, \mu_t) dW_t, \quad Y_0^\mu = \xi, \quad (4)$$

where $\mu = (\mu_t)_{t \in [0, T]}$. Notice that if μ is fixed, then (4) has a unique strong solution [4, 5]. You have to use some fixed point theorem argument on F to yield the wellposedness of (3). In order to do so, you should exploit the definition of Wasserstein distance and bound it with a particular coupling.

2. You can assume that $\sup_{s \in [0, t]} \mathcal{W}_2(\mu_s, \eta_s) \leq \mathcal{W}_2(\mu, \eta)$, where $\mathcal{W}_2(\mu, \eta)$ is the Wasserstein distance defined on $\mathcal{P}_2(C([0, t]; \mathbb{R}^d))$ and $C([0, t]; \mathbb{R}^d)$ is endowed with the uniform convergence metric.
3. It will be needed to show that

$$\mathbb{E} \left[\sup_{s \in [0, t]} \|Y_s^\mu - Y_s^\eta\|^2 \right] \leq C \int_0^t \mathcal{W}_2^2(\mu_r, \eta_r) dr \quad (5)$$

for $C > 0$ and to do so, consider also using Gronwall's lemma.

4. Use (5) and Gronwall's lemma again, and infer from the usual Picard iteration argument for standard SDEs that there exists a unique fixed point.

Report only the steps that are substantially different from existence and uniqueness result for standard SDEs under Assumption 1 ([4, 5]).

- **(Q3)** We go back to the n -particle SDE system (1).

1. Prove the well-posedness of (1) under Assumption 1.

Hint: Notice that (1) can be rewritten in the Itô SDE form and verify that the drift and the diffusion satisfy the standard assumptions of well-posedness for general SDEs.

2. Prove the propagation of chaos for (1), i.e.

$$\lim_{n \rightarrow \infty} \mathbb{E}[\mathcal{W}_2^2(\mu^n, \mu)] = 0,$$

where $\mu = \mathcal{L}(Y)$ defined in (3). Is the result coherent with **(Q1)** ?

Hint:

- (a) First, compare (1) with n copies of (3), namely $\{Y^i\}_{i=1,\dots,n}$, driven by the same n -independent Brownian motions W_t^i and with the same initial data ξ^i . Defining the empirical measure $\eta^n = \frac{1}{N} \sum_{i=1}^n \delta_{Y_t^i}$, you can bound the error in the following way

$$\mathbb{E}[\mathcal{W}_2^2(\mu^n, \mu)] \leq 2\mathbb{E}[\mathcal{W}_2^2(\mu^n, \eta^n)] + 2\mathbb{E}[\mathcal{W}_2^2(\eta^n, \mu)] \quad (6)$$

- (b) For estimating the first term on the right hand side, the proof will follow very closely to the one of **(Q2)**, comparing (1) with $\{Y^i\}_{i=1,\dots,n}$.

- (c) To conclude, use the Gronwall's lemma and the following result.

Theorem 1. *Suppose $\{Y^i\}$ are i.i.d $\mathbb{R}^d$ -valued random variables having law μ . Consider also the empirical measure $\eta^n = \frac{1}{N} \sum_{i=1}^n \delta_{Y_t^i}$. If $\mu \in \mathcal{P}_p(\mathbb{R}^d)$, with $p \geq 1$, then*

$$\lim_{n \rightarrow \infty} \mathbb{E}[\mathcal{W}_p^p(\eta^n, \mu)] = 0.$$

- **(Q4)** Suppose that the law of Y_t (from (3)) has a density $p(x, t) \in C^{2,1}(\mathbb{R}^d \times (0, +\infty))$, whereas the law of ξ has density p_0 . Then p is a solution of the initial value problem for the Fokker-Planck equation (in the strong form)

$$\begin{aligned} \frac{\partial p}{\partial t} &= \mathcal{L}^* p, \quad \text{in } (x, t) \in \mathbb{R}^d \times (0, +\infty), \\ p(x, 0) &= p_0, \quad \text{in } x \in \mathbb{R}^d, \end{aligned} \quad (7)$$

where $\mathcal{L}^*$ is the adjoint operator of the generator $\mathcal{L}$ of the SDE in (3). One has a weak version of (7), too. For more details on the whole topic, see for example [6].

1. Derive the generator of (3) and the Fokker-Planck Equation (FPE) associated to (3). What is the main difference of this FPE with respect to the one of a standard SDE?

Hint: the term μ_t^Y should be treated as fixed when using Itô's formula.

2. What is a natural boundary condition to assume for the well-posedness of the PDE (7), i.e. $\lim_{|x| \rightarrow +\infty} p(x, t)$ for all t ?

- **(Q5)** Starting from this point, we will analyze some numerical aspects inherent to the McKean Vlasov equations.

Let us consider the following scalar Black-Scholes model with expectation:

$$dS(t) = (IS(t) - \mathbb{E}[S(t)])dt + \lambda(S(t) + \mathbb{E}[S(t)])dW_t, \quad (8)$$

where W_t is a one-dimensional Brownian motion, $I = 1.1$, $\lambda = 0.1$, $S_0 = 1$, $T = 1$, for all the paths.

1. Implement the Euler Scheme described in [2, Equation 2.3], whose m -step for a general MV-SDE in dimension 1 reads as

$$\begin{aligned} X_{t_{m+1}}^{n,i} &= X_{t_m}^{n,i} + a\left(X_{t_m}^{n,i}, \mu_{t_m}^n\right) \Delta t + b\left(X_{t_m}^{n,i}, \mu_{t_m}^n\right) \Delta W_t^i, \quad X_0^{n,i} = \xi^i, \quad i = 1, 2, \dots, n \\ \mu_{t_m}^n &= \frac{1}{n} \sum_{k=1}^n \delta_{X_{t_m}^{n,k}}, \end{aligned} \quad (9)$$

for a uniform step size $\Delta t = t_{m+1} - t_m$ for $m = 1, \dots, M$ with $\Delta t = \frac{T}{M}$ and ΔW_t usual Brownian increment. Estimate numerically the strong rate of convergence in the number of particles fixed the timestep Δt (you can simulate true solutions with the same method with a number of particles $N \approx 10^5$ or 10^6) and the rate in Δt fixed the number of particles (you can simulate true solution with the same method with a timestep $\Delta t \approx 10^{-5}$ or 10^{-6} ; in case your laptop does not sustain simulations with these data, you can tune them.)

2. Discuss the overall computational cost of the Algorithm based on the Euler Scheme considered in point **(Q5)-1** to achieve a given tolerance Tol on the strong error.
3. Implement the Milstein scheme described by [1, Equation 2.11], which for a general MV-SDE in dimension 1 reads

$$\begin{aligned}
X_{t_{m+1}}^{n,i} &= X_{t_m}^{n,i} + a(X_{t_m}^{n,i}, \mu_{t_m}^n) \Delta t + b(X_{t_m}^{n,i}, \mu_{t_m}^n) \Delta W_t^i \\
&\quad + \partial_x b(X_{t_m}^{n,i}, \mu_{t_m}^n) b(X_{t_m}^{n,i}, \mu_{t_m}^n) \left(\frac{(\Delta W_t^i)^2}{2} - \frac{\Delta t}{2} \right), \quad X_0^{n,i} = \xi^i, \quad i = 1, 2, \dots, n \\
\mu_{t_m}^n &= \frac{1}{n} \sum_{k=1}^n \delta_{X_{t_m}^{n,k}},
\end{aligned} \tag{10}$$

where ∂_x denotes the derivative with respect to the first variable of b , and answer the same question of **(Q5)-1** (in case your laptop does not sustain simulations with these data, you can tune them.).

4. Discuss the whole computational cost of the Algorithm considered in point **(Q5)-3** to achieve a given tolerance Tol on the strong error.
- **(Q6)** Suppose to approximate the FPE described by Equation (7) via a finite difference method of your choice with mesh size $h = \frac{1}{N}$ and call this solution $\hat{\mu}^n$. In this case, you will have to truncate the domain and enforce some boundary condition (e.g. homogeneous Dirichlet). Consider to use $\hat{\mu}^n$ instead of the empirical measure in the Euler scheme in (9) applied to (3). Discuss (briefly) the full approximation algorithm, the full computational cost and how the rate of convergence changes.
 - **(Q7)** Well-posedness of equation (3) can be proven also in the case of local Lipschitz drift and diffusion, satisfying a weak monotonicity condition. However, its numerical simulation can present some problems. Let us consider the following equation

$$dZ_t = \left(2\alpha \mathbb{E}[Z_t] + \frac{\beta^2 Z_t}{2} - Z_t^3 \right) dt + \gamma Z_t dW_t, \tag{11}$$

where W_t is a one-dimensional Brownian motion and $Z_t(\omega) \in \mathbb{R}$ for all ω .

1. Consider the particle system method in (9). Suppose that $Z_0 = 1$, $\alpha = 0.25$, $\beta = 2$, $\gamma = 1.5$, $T = 3$. Simulate (11) with a number of particles $N = 2000$ and time mesh $\Delta t = 0.05$ and plot all the paths in one graph.
 2. Repeat the same computation of **(Q7)-1** but with a number of particles $N = 2500, 5000, 7500, 10000, 15000$. Comment the results.
 3. Repeat the same computation of **(Q7)-1** but with diffusion noise $\gamma = 0.5, 1, 1.5, 2, 2.5$. Comment the results.
 4. The effect that you should have noticed in the previous points is called *particle corruption*. Can you explain why it appears? Actually, this kind of effect can emerge also in the case of standard SDEs. However, the mean-square error for a standard SDE is usually “less affected” than the one for a McKean-Vlasov SDE. Can you explain why?
- **(Q8)** The authors in [3] suggests a split-step method for McKean-Vlasov SDEs to deal with equations like (11). The m -step reads as

$$Y_{m+1}^{n,i} = X_{t_m}^{n,i} + v(Y_{m+1}^{n,i}, \mu_{t_m}^{X,n}) \Delta t \tag{12}$$

$$X_{t_{m+1}}^{n,i} = Y_{m+1}^{n,i} + u(Y_{m+1}^{n,i}, \mu_{t_{m+1}}^{Y,n}) \Delta t + b(Y_{m+1}^{n,i}, \mu_{t_{m+1}}^{Y,n}) \Delta W_t^i, \quad X_0^{n,i} = \xi^i, \quad i = 1, 2, \dots, n \tag{13}$$

$$\mu_{t_m}^{X,n} = \frac{1}{n} \sum_{k=1}^n \delta_{X_{t_m}^{n,k}}, \quad \mu_{t_m}^{Y,n} = \frac{1}{n} \sum_{k=1}^n \delta_{Y_{t_m}^{n,k}}, \tag{14}$$

where the drift a has been decomposed as $a := v + u$, where u satisfies Assumption 1, while v is locally Lipschitz in space and globally in the measure variable, and satisfies the one-sided Lipschitz property: $\exists L_v > 0$ such that $\langle x - y, v(x, \mu) - v(y, \mu) \rangle \leq L_v |x - y|^2$, for all $x, y \in \mathbb{R}^d$, $\mu \in \mathcal{P}_2(\mathbb{R}^d)$.

1. Implement the method with the same data of **(Q7)-2** and comments the results.

Hint: For the implicit step (12) you can use an iterative method of your choice, for example `scipy.optimize.newton` for Python or `newtons_method` for Matlab.

2. Implement the method with the same data of **(Q7)-3** and comments the results.
3. What is the total computational cost of the split-step algorithm? Discuss possible improvements in the implementation.

References

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