Project for the course "Numerical Integration for Stochastic Differential Equations"

Modeling asset pricing via SDEs

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In finance, an asset is a resource with economic value that an investor owns with the expectation that it will give a future benefit. Specifically, the price of an asset $S(t)$ is a quantity whose current value is known, but the future one is not. Its evolution depends on the expected rate of return $I(t)$, i.e., the estimated profit that an investor expects to achieve, and on the volatility $\sigma(t)$, which is a measure of the risk in the market. By their nature, $I(t)$ and $\sigma(t)$ should be nonnegative. All of these quantities can vary in time and are supposed to be random. Therefore, one can see S, I, and σ as stochastic processes.

Let us consider the following stochastic basis $(\Omega, \mathcal{A}, \mathbb{P}; (\mathcal{F}_t)_{t\geq 0})$ and assume that $S(t)$ is described by the following one-dimensional equation

$$\mathrm{d}S(t) = I(t)S(t)\mathrm{d}t + \sigma(t)S(t)\mathrm{d}W(t), \tag{1}$$

where W is a one-dimensional Brownian-motion adapted to the filtration $(\mathcal{F}_t)_{t\geq 0}$. We are interested in knowing the stochastic properties of (1) and numerically quantifying the price of an asset.

- **(Q1)** In order to determine the price of an asset over time, we need to solve the SDE (1).
  1. Derive a closed form expression for the solution of (1) assuming constant $I(t) \equiv I$, $\sigma(t) \equiv \sigma$, with $\sigma, I \in \mathbb{R}_+$.
  2. Show that the process $G(t) = e^{\sigma W(t)}$ is a submartingale, but not a martingale.
  3. Find for which values of I $S(t)$ in **(Q1)-1** is a martingale.
  4. Find the solution of (1) in closed form assuming generic $I(t)$, $\sigma(t)$.
- **(Q2)** Assume that the interest rate $I(t)$ is stochastic and it is determined by some Itô differential equation. Consider the following equation

  $$\mathrm{d}I(t) = (a - bI(t))\mathrm{d}t + c\mathrm{d}W_t^{(2)}, \tag{2}$$

  with $I(0) > 0$, $a, b, c > 0$ and $W_t^{(2)}$ a one-dimensional Brownian motion adapted to $(\mathcal{F}_t)_{t\geq 0}$, independent of W_t.
  1. Find a closed form expression for the solution of (2).
  2. What is the distribution of $I(t)$? Compute the probability that $I(t) < 0$ with $a = b = c = 1$ for $t = 1, 5, 10, 15, 20$. Could this behavior be a problem from the financial point of view?
  3. Let us consider the following alternative interest model

     $$\mathrm{d}I(t) = (a - bI(t))\mathrm{d}t + c\sqrt{I(t)}\mathrm{d}W_t^{(2)}, \tag{3}$$

     It is known that (3) admits a unique strong solution I. Prove that $I(t)$ is nonnegative for all $t \in \mathbb{R}$. *Hint:* you can consider the following ausiliary process (which also have a unique strong solution)

     $$\mathrm{d}I(t) = (a - bI(t))\mathrm{d}t + c\sqrt{(I(t) \vee 0)}\mathrm{d}W_t^{(2)}, \tag{4}$$

     and the stopping time $\tau_\epsilon = \inf\{t \ : \ I(t) = -\varepsilon\}$ for any ε small enough and show that $\mathbb{P}(\tau_\epsilon < \infty) = 0$.
  4. (2) is a mean-reverting process. What does it mean? If $I(0) = \frac{a}{b}$, what can we say about $\mathbb{E}[I(t)]$? And if $I(0) \neq \frac{a}{b}$? Is (3) mean-reverting?
- **(Q3)** Let us suppose that $\sigma = \sqrt{v(t)}$, where $v(t)$ is non-negative process defined as $v(t) = \tilde{\sigma}^2(t)$ with

  $$\mathrm{d}\tilde{\sigma}(t) = -\lambda\tilde{\sigma}(t)\mathrm{d}t + f\mathrm{d}W_t^{(3)}, \tag{5}$$

  where $\lambda, f > 0$ and $W_t^{(3)}$ is a one-dimensional Brownian motion adapted to $(\mathcal{F}_t)_{t\geq 0}$, independent of W_t and $W_t^{(2)}$ (notice that in this model $\sigma(t) = |\tilde{\sigma}(t)|$).

1. Derive a closed form expression for the solution of (5).
2. The process $\tilde{\sigma}(t)$ of (5) has the property of being ergodic, i.e., its distribution μ_t tends for $t \to \infty$ to an invariant measure, which we denote by μ_∞ . Such limit distribution admits a probability density function ρ_∞ with respect to the Lebesgue measure on $\mathbb{R}$. Derive the invariant measure of (5) and verify that the corresponding probability density function satisfies the Fokker-Planck equation.
3. Show that the following equation holds for the square volatility process $v(t) = \tilde{\sigma}^2(t)$ with suitable values of k, μ, η

$$\mathrm{d}v(t) = k\,[\mu - v(t)]\,\mathrm{d}t + \eta\sqrt{v(t)}\mathrm{d}W_t^{(3)}, \tag{6}$$

and specify k, μ, η as functions of λ and f. Briefly argument whether (6) is a good model from the financial point of view or not.

- **(Q4)** Suppose to approximate (1) with $I(t) = I$ and $\sigma(t) = \sigma$ with the Euler-Maruyama method using a uniform time-step Δt, and denote by $\hat{S}(t)$ its numerical solution (continuously interpolated process) on $[0, T]$.

1. Which is the order of convergence of $\sqrt{\mathbb{E}[\sup_{0 \le t \le T} (S(t) - \hat{S}(t))^2]}$?
2. Suppose now that $I(t)$ in (1) is described by (2) and consider an approximation of $I(t)$ by the Euler-Maruyama method with a uniform time-step Δt (the same as in 1.). Which is the order of convergence of $\sqrt{\mathbb{E}[\sup_{0 \le t \le T} (S(t) - \hat{S}(t))^2]}$? Provide a rigorous argument.

   Hint: To simplify your argument, you can consider the following modification of (1)

$$dS = \varphi_M(I(t))S(t)dt + \varphi_M(\sigma(t))S(t)dW_t,$$

   where $\varphi_M(x) = \max\{-M, \min\{M, x\}\}$ and $M > 0$ a large enough constant. Notice that φ_M is bounded and Lipschitz function with Lipschitz constant 1. You might need to derive uniform bounds on L^p norms ($p \ge 2$) of the exact solution $\mathbb{E}[|S(t)|^p]$ and numerical one $\mathbb{E}[|\hat{S}(t)|^p]$.
3. Suppose that $I(t)$ and $\sigma(t)$ in (1) are described by (2) and (5), respectively. Consider a discretization of both of them by the Euler-Maruyama method with a uniform time-step Δt (the same as 1.). Which is the order of convergence of $\sqrt{\mathbb{E}[\sup_{0 \le t \le T} (S(t) - \hat{S}(t))^2]}$? Which is the order of convergence of the Euler-Maruyama method applied to $I(t)$ and $\sigma(t)$? Provide rigorous arguments.
4. Let us assume that $a = 0.1$, $b = c = 0.5$, $d = f = 1$. Consider $T = 2$, and simulate the Brownian motion $W_t, W_t^{(2)}, W_t^{(3)}$ with a uniform time-step $\mathrm{d}t = 10^{-5}$ (i.e. 200001 points) and $M = 100$ paths. Moreover, assume $S(0) = 2$, $I(0) = 0.5$, and $\sigma(0) = 2$ for all paths. Compute the solution $S(t)$ of (1) with interest rate $I(t)$ and $\sigma(t)$ defined by the closed formulae obtained in **(Q3)**-1. and **(Q4)**-1 (you can approximate the integrals present therein with a left point quadrature formula as in the Euler method, and using the generated paths of $W_t, W_t^{(2)}, W_t^{(3)}$), and their Euler-Maruyama approximation for uniform $\Delta t = 5 \cdot 10^{-4}, 10^{-3}, 4 \cdot 10^{-3}, 8 \cdot 10^{-3}, 10^{-2}$. Plot the error $\sqrt{\mathbb{E}[(S(T) - \hat{S}(T))^2]}$ (you can estimate the expectation with a Monte-Carlo procedure). What order do you observe? Is it consistent with your conclusions of the previous points?

   (If your machine does not sustain computations with these time-steps, you can change them accordingly.)

- **(Q5)** Assume that in (1), $I(t) = I$ and $\sigma(t) = \sigma$ are <u>real</u> constants, and also assume $S(0) = 2$. Consider $T = 5$ and $M = 1000$ paths.

1. Show that

$$\lim_{t \to +\infty} \frac{1}{t} \log(S(t)) = I - \frac{\sigma^2}{2}, \text{ a.s. if } I \neq \frac{\sigma^2}{2} \tag{7}$$

   and

$$\limsup_{t \to +\infty} \frac{\log(S(t)}{\sqrt{2t \log\log t}} = \sigma \text{ and } \liminf_{t \to +\infty} \frac{\log(S(t)}{\sqrt{2t \log\log t}} = -\sigma, \text{ a.s. if } I = \frac{\sigma^2}{2} \tag{8}$$

   Deduce a condition of mean-square stability of the price S, i.e. $\lim_{t \to +\infty} \mathbb{E}[|S_t|^2] = 0$.

2. Assume $\sigma = 0.12$. For $\Delta t = 0.01, 0.02, 0.05, 0.1$, plot $\mathbb{E}[|S(t)|^2]$ for the interest rates $I = 0.002, 0.006, 0.0072, 0.012, 0.024$ (*you can approximate it with a forward Euler method*) and comment the results. Is this performance a good asset behavior? Comment.
3. Suppose to apply the stochastic-θ method to (1), denoting its solution by $(S_n^\theta)_n$. Discuss the mean-square stability of the stochastic-θ method applied to (1), i.e. $\lim_n \mathbb{E}[|S_n^\theta|^2] = 0$, and derive sufficient conditions of mean-square stability for $\theta = 0, \frac{1}{2}, 1$. Verify numerically the conditions for $\theta = 0, \frac{1}{2}, 1$, with the data of **(Q5)-2**.
4. Assume now that $\sigma(t)$ is no more constant and satisfies equation (5), in **(Q3)**. Consider an approximation of $\sigma(t)$ by the Euler-Maruyama method, denoting its solution by $(\sigma_n)_n$. Find a condition on Δt so that there exists a constant $M > 0$ such that $\mathbb{E}[|\sigma_n|^2] \leq M$ for all n. Do we have mean-square stability of $(\sigma_n)_n$ under such condition on Δt? Verify these properties numerically for $f = 0.2, \lambda = 0.1, T = 5$, and $\sigma_0 = 1$.

- **(Q6)** Suppose to have a *basket of options*, i.e. a collection of multiple financial securities, composed only by assets whose prices are described by the following system of SDEs

$$\mathrm{d}S(t) = IS(t)\mathrm{d}t + \sigma K(S(t) \circ \mathrm{d}W_t), \tag{9}$$

where, $S(t, \omega) \in \mathbb{R}^n$ fixed $\omega \in \Omega$, $(S(t) \circ \mathrm{d}W_t)$ denote the Hadamard product between $S(t)$ and $\mathrm{d}W_t$, i.e. $A \circ B = [a_{ij}b_{ij}]_{ij}$ for $A, B \in \mathbb{R}^{m,n}$, $K \in \mathbb{R}^{n\times n}$ is a symmetric positive definite matrix, I is a symmetric positive definite matrix, σ is a nonnegative constants and W is a n-dimensional Brownian motion adapted to $(\mathcal{F}_t)_{t\geq 0}$.

1. Derive an equation for $G(t) := \mathbb{E}[S(t)S(t)^\top]$.
2. Assume $T = 1$, $I = 0.01 \cdot I_{d\times d}$, $K = RR^\top$, with R a matrix whose entries are sampled from a $\mathrm{Unif}(0, 1)$ r.v., and $\sigma = 10^{-3}$.
   Approximate (9) with a Euler-Maruyama method. Denote $(S_n)_n$ the approximate solution and compute the approximate matrix $(\mathbb{E}[S_n S_n^\top])_n$ at discrete times $t_n = n\Delta t$, approximating the expectation $\mathbb{E}$ with a Monte-Carlo method $\hat{\mathbb{E}}$. Compute the error

$$\sqrt{\sum_{n=0}^{N} \|\hat{\mathbb{E}}[S_n S_n^\top] - G(t_n)\|_F^2 \Delta t} \tag{10}$$

   for $\Delta t = 0.001, 0.01, 0.05, 0.1$, $M = 100$, $d = 5$. Which are the sources of errors in computing (10)? Repeat the same computatations but with $M = 2000$ and comment the results.

- **(Q7)** Consider equation (6). Let $k = 1$, $\mu = 1$, $v(0) = 1$ for all the $M = 500$ paths.

1. In [1], a Euler-type method based on the Lamperti transformation $Y_t = \sqrt{v_t}$ is proposed, obtaining the accuracy result of Theorem 1.1 therein. Derive the same discretization scheme for (6) and show with a simulation that Theorem 1.1 holds, fixing $\eta = 0.5, 1.5, 1.75$. Compare the convergence rate with the one of the Stochastic θ-method with $\theta = 1$ applied to the same problem (you can compute the true solution using [1] and a very fine mesh (at least $\Delta t \approx 10^{-5}$)). Comment the results.
2. Why can't we use the formula obtained in **(Q3)-1** in point **(Q7)-1**?

- **(Q8)** The theory of asset pricing is the standard tool in the evaluation of *options*. An option is a contract between two parties, a "writer" and a "holder". Both sides agree that a fixed time $t = T$, called the *expire date*, the writer will pay an amount of money to the holder, called the *payoff*, whereas the contract is assumed to be stipulated at time $t = 0$. The value of the *payoff* is determined by the behavior of the price $S(t)$ of the chosen asset in the interval of time $[0, T]$. On the other hand, the holder does have to pay a small amount, called "premium", to the writer at time $t = 0$ by contract. For example, let us consider a "European call" option, which is defined by the following payoff function

$$P(S(T)) = \max\{S(T) - K, 0\}, \tag{11}$$

where $K > 0$ is called the exercise price decided in the option. If $S(T) > K$, the holder of the option will receive a positive payoff amount $P(S(T))$ from the writer, otherwise no money will be received. Therefore, option pricing is essentially betting at time $t = 0$ about what will be the value $P(S(T))$, i.e. the random variable $S(T)$, so that the holder and writer can deal the best option for themselves.

Suppose that now $I(t)$ and $\sigma(t)$ are real positive constants. Consider $S(0)$ the initial condition of the asset.

Under the no-arbitrage assumption, the fair value of the premium is given by

$$C(S_0, T) = e^{-rt}\mathbb{E}[P(S(T))]. \tag{12}$$

where r is the risk free interest rate. A closed formula for (12) is

$$C(s, t) = s\mathcal{N}(x_1) - Ke^{-rt}\mathcal{N}(x_2) \tag{13}$$

where

$$x_1 = \frac{log(s/K) + (r + \frac{1}{2}\sigma^2 t)}{\sigma\sqrt{T}}, \quad x_2 = \frac{log(s/K) + (r - \frac{1}{2}\sigma^2 t)}{\sigma\sqrt{T}}$$

and $\mathcal{N}$ denotes the standard normal cumulative distribution function. Approximate (12) with the Euler-Maruyama algorithm, where I is replace by r in the dynamics (1), computing the expectation in (12) by Monte-Carlo with $M = 100$ realizations, considering $S(0) = 8$, $K = 10$, $r = 0.05$, $\sigma = 0.5$ and $T = 1$. Plot the error between the approximation and true solution at time T commenting the results. Is the result consistent with the theory?

References

[1] Steffen Dereich, Andreas Neuenkirch, and Lukasz Szpruch. "An Euler-type method for the strong approximation of the Cox–Ingersoll–Ross process". In: *Proceedings of the royal society A: mathematical, physical and engineering sciences* 468.2140 (2012), pp. 1105–1115.