Series 8 - November 13, 2024

Exercise 1.

Let $\{W_t\}_t$ be an m-dimensional Brownian motion. We recall the definitions

$$I_n(k,l) = \int_{t_n}^{t_{n+1}} \int_{t_n}^{s} dW_u^k dW_s^l, \quad I_n(k,l,j) = \int_{t_n}^{t_{n+1}} \int_{t_n}^{s} \int_{t_n}^{u} dW_r^k dW_u^l dW_s^j,$$

with the convention $W_t^0 = t$ (hence $dW_t^0 = dt$). Show that

1) $I_n(0,0) = \frac{1}{2}\Delta t^2$,
2) $I_n(0,k) = \Delta t \Delta W_n^k - I_n(k,0)$
3) $I_n(k,0) = \frac{1}{2}\Delta t(\Delta W_n^k + \frac{1}{\sqrt{3}} Z_k)$ with $Z^k \sim N(0,\Delta t)$ independent of ΔW_n^k.
4) $I_n(k,k,k) = \frac{1}{2}(\frac{1}{3}(\Delta W_n^k)^2 - \Delta t)\Delta W_n^k$

Solution

1) $I_n(0,0) = \int_{t_n}^{t_{n+1}} \int_{t_n}^{s} duds = \int_{t_n}^{t_{n+1}} (s-t_n)ds = \frac{1}{2}\Delta t^2$,
2) Notice that

$$\begin{aligned} I_n(k,0) &= \int_{t_n}^{t_{n+1}} \int_{t_n}^{s} dW_u^k ds = \int_{t_n}^{t_{n+1}} (W_s^k - W_{t_n}^k) ds \\ &= \int_{t_n}^{t_{n+1}} W_s^k ds - W_{t_n}^k \Delta t \end{aligned} \tag{1.1}$$

Therefore, one has

$$\begin{aligned} I_n(0,k) &= \int_{t_n}^{t_{n+1}} \int_{t_n}^{s} du dW_s^k = \int_{t_n}^{t_{n+1}} (s-t_n) dW_s^k \\ &= \int_{t_n}^{t_{n+1}} s dW_s^k - t_n \Delta W_n^k \\ &= \int_{t_n}^{t_{n+1}} d(sW_s^k) - \int_{t_n}^{t_{n+1}} W_s^k ds - t_n \Delta W_n^k \\ &= t_{n+1} W_{n+1}^k - t_n W_{t_n}^k - \int_{t_n}^{t_{n+1}} W_s^k ds - t_n \Delta W_n^k \\ &= t_{n+1} W_{t_{n+1}}^k - t_{n+1} W_{t_n}^k + t_{n+1} W_{t_n}^k + -t_n W_{t_n}^k - \int_{t_n}^{t_{n+1}} W_s^k ds - t_n \Delta W_n^k \\ &= \Delta t \Delta W_n^k - \int_{t_n}^{t_{n+1}} W_s^k ds + W_{t_n}^k \Delta t \\ &= \Delta t \Delta W_n^k - I_n(k,0). \end{aligned} \tag{1.2}$$

3) We have

$$\mathbb{E}(I_n(k,0)) = \mathbb{E}\Big(\int_{t_n}^{t_{n+1}}\int_{t_n}^{s_1} \mathrm{d}W^k_{s_2}\mathrm{d}s_1\Big) = \int_{t_n}^{t_{n+1}}\mathbb{E}\Big(\int_{t_n}^{s_1}\mathrm{d}W^k_{s_2}\Big)\mathrm{d}s_1 = 0,$$

and

$$\mathbb{E}(\Delta W_n I_n(k,0)) = \int_{t_n}^{t_{n+1}}\mathbb{E}(\Delta W_n(W^k_{s_1} - W(t_n)))\mathrm{d}s_1 = \int_{t_n}^{t_{n+1}}(s-t_n)\mathrm{d}s_1 = \frac{(\Delta t)^2}{2}.$$

Then, by Itô isometry we get

$$\mathbb{E}(I_n(0,k)^2) = \mathbb{E}\left(\left(\int_{t_n}^{t_{n+1}} s_1\mathrm{d}W^k_{s_1}\right)^2\right) = \int_{t_n}^{t_{n+1}}(s-t_n)^2\mathrm{d}s_1 = \frac{(\Delta t)^3}{3}.$$

Moreover, due to $I_n(0,k) = \Delta t\Delta W_n - I_n(k,0)$ we have

$$\mathbb{E}(I_n(0,k)\Delta W_n) = \Delta t\mathbb{E}(\Delta W_n^2) - \mathbb{E}(\Delta W_n I_n(k,0)) = (\Delta t)^2 - \frac{(\Delta t)^2}{2} = \frac{(\Delta t)^2}{2}.$$

Therefore, we obtain

$$\begin{aligned}\mathbb{E}(I_n(k,0)^2) &= \mathbb{E}((\Delta t\Delta W_n - I_n(0,k))^2) = (\Delta t)^2\mathbb{E}(\Delta W_n^2) + \mathbb{E}(I_n(0,k)^2) - 2\Delta t\mathbb{E}(I_n(0,k)\Delta W_n)\\ &= (\Delta t)^3 + \frac{(\Delta t)^3}{3} - (\Delta t)^3 = \frac{(\Delta t)^3}{3}.\end{aligned}$$

Consider now the joint process $(\Delta W, I_n(k,0))$, which is distributed as a bivariate Gaussian with zero mean and covariance matrix

$$\Sigma = \begin{pmatrix}\Delta t & (\Delta t)^2/2\\ (\Delta t)^2/2 & (\Delta t)^3/3\end{pmatrix}.$$

In order to find the thesis, we want to compute the Cholesky factor of the covariance Σ, i.e., the lower triangular matrix L such that $\Sigma = LL^\top$, and then the joint process $(\Delta W, I_n(k,0))$ as

$$\begin{pmatrix}\Delta W\\ I_n(k,0)\end{pmatrix} = L\begin{pmatrix}V_1\\ V_2\end{pmatrix},$$

where $V_1, V_2 \sim \mathcal{N}(0,1)$ are two independent random variables, so that expliciting $I_n(k,0)$ will give us the wanted expression. It is possible to show that L is given by

$$L = \begin{pmatrix}\sqrt{(\Delta t)} & 0,\\ (\Delta t)^{3/2}/2 & \sqrt{3}(\Delta t)^{3/2}/6\end{pmatrix},$$

which yields

$$\begin{pmatrix}\Delta W\\ I_n(k,0)\end{pmatrix} = L\begin{pmatrix}V_1\\ V_2\end{pmatrix} = \begin{pmatrix}\sqrt{(\Delta t)}V_1\\ \frac{1}{2}(\Delta t)^{3/2}(V_1 + \frac{1}{\sqrt{3}}V_2)\end{pmatrix}.$$

Therefore

$$I_n(k,0) = \frac{1}{2}(\Delta t)(\Delta t V_1 + \frac{1}{\sqrt{3}}(\Delta t)^{1/2}V_2) = \frac{1}{2}\Delta t(\Delta W^k_n + \frac{1}{\sqrt{3}}Z_k)$$

with $Z^k \sim N(0,\Delta t)$ independent of ΔW^k_n.

4) We set $\tilde{W}^k_s = W^k_s - W^k_{t_n}$ for $s \in [t_n, t_{n+1}]$. Notice that $d\tilde{W}^k_s = dW^k_s$. We first have

$$\begin{aligned}I_n(k,k,k) &= \int_{t_n}^{t_{n+1}}\int_{t_n}^{s_1}\int_{t_n}^{s_2} d\tilde{W}^k_{s_3}d\tilde{W}^k_{s_2}d\tilde{W}^k_{s_1}\\ &= \int_{t_n}^{t_{n+1}}\int_{t_n}^{s_1}\tilde{W}^k_{s_2}d\tilde{W}^k_{s_2}d\tilde{W}^k_{s_1}\\ &= \frac{1}{2}\int_{t_n}^{t_{n+1}}(\tilde{W}^k_{s_1})^2 d\tilde{W}^k_{s_1} - \frac{1}{2}\int_{t_n}^{t_{n+1}}(s_1 - t_n)d\tilde{W}^k_{s_1}.\end{aligned} \tag{1.3}$$

Applying the stochastic integration by parts we obtain

$$\frac{1}{3}(\tilde{W}^k_{t_{n+1}})^3 = \int_{t_n}^{t_{n+1}} (\tilde{W}^k_s)^2 d\tilde{W}^k_s + \int_{t_n}^{t_{n+1}} \tilde{W}^k_s ds, \tag{1.4}$$

which can be written in the differential form as

$$\frac{1}{3}d(\tilde{W}^k_s)^3 = (\tilde{W}^k_s)^2 d\tilde{W}^k_s + \tilde{W}^k_s ds. \tag{1.5}$$

Finally, by previous computations we get

$$\int_{t_n}^{t_{n+1}} (s - t_n) d\tilde{W}^k_s = \Delta t \Delta W^k_n - \int_{t_n}^{t_{n+1}} \tilde{W}^k_s ds. \tag{1.6}$$

Putting all together one finds that $I_n(k,k,k) = \frac{1}{3!}((\Delta W^k_n)^2 - 3\Delta t)\Delta W^k_n$, which is equivalent to the thesis.

Exercise 2.

Let $W_1, \ldots, W_m$ be m independent one-dimensional Brownian motions in $[0,T]$. Consider a uniform partition $P = \{0 = t_0 < t_1 < \ldots < t_N = T\}$ of size Δt. We recall that

$$I_n(i_p, \ldots, i_1) := \int_{t_n}^{t_{n+1}} \int_{t_n}^{s_1} \ldots \int_{t_n}^{s_{p-1}} \mathrm{d}W^{i_p}_{s_p} \ldots \mathrm{d}W^{i_1}_{s_1}$$

where for $\mathrm{d}W^{i_j}_s = 0$if $i_j = 0$ For example, we have

$$I_n(1,2,0) = \int_{t_n}^{t_{n+1}} \int_{t_n}^{s_1} \int_{t_n}^{s_2} \mathrm{d}W^1_{s_3} \mathrm{d}W^2_{s_2} \mathrm{d}s_1.$$

Show that:

i) $\mathbb{E}[I_n(i_p, \ldots, i_1)] = 0$ if there exists $\hat{j} \in \{1, \ldots, p\}$ such that $i_{\hat{j}} \neq 0$.

ii) $\mathbb{E}[I_n(i_p, \ldots, i_1)] = \mathcal{O}((\Delta t)^p)$ if $i_j = 0$ for all $j \in \{1, \ldots, p\}$.
Hint. Use the Cauchy formula for repeated integration.

iii) $\mathbb{E}[I^2_n(i_p, \ldots, i_1)]^{1/2} = \mathcal{O}((\Delta t)^q)$ where $q = \sum_{j=1}^p \frac{2-\min\{1,i_j\}}{2}$.
Hint. Prove it by induction.

iv) $\mathbb{E}[|I_n(i_p, \ldots, i_1)|] = \mathcal{O}((\Delta t)^q)$ where $q = \sum_{j=1}^p \frac{2-\min\{1,i_j\}}{2}$.

Solution

The first point is trivial, since for any $G \in \mathrm{M}^2(0,T)$

$$\mathbb{E}\Big(\int_{t_n}^{t_{n+1}} G \mathrm{d}W^{i_j}_{s_j}\Big) = 0.$$

The second point is a standard result (Cauchy formula for iterated integral). The third point is proved by induction on p. For $p = 1$, we have the two following possible cases

$$\mathbb{E}\Big(\big(\int_{t_n}^{t_{n+1}} \mathrm{d}s_1\big)^2\Big) = (\Delta t)^2, \quad \mathbb{E}\Big(\big(\int_{t_n}^{t_{n+1}} \mathrm{d}W^{i_1}_{s_1}\big)^2\Big) = \mathbb{E}\big(\int_{t_n}^{t_{n+1}} \mathrm{d}s_1\big) = \Delta t,$$

and therefore the assertion holds. Assume now the assertion iii) to be true for $p-1$ and consider $I_n(i_p,\dots,i_1)$,

$$I_n(i_p,\dots,i_1)=\int_{t_n}^{t_{n+1}}\int_{t_n}^{s_1}\dots\int_{t_n}^{s_{p-1}}\mathrm{d}W_{s_p}^{i_p}\dots\mathrm{d}W_{s_1}^{i_1}=\int_{t_n}^{t_{n+1}}I_n(i_p,\dots,i_2)[s_1]\mathrm{d}W_{s_1}^{i_1},$$

where we denoted

$$I_n(i_p,\dots,i_2)[s_1]=\int_{t_n}^{s_1}\dots\int_{t_n}^{s_{p-1}}\mathrm{d}W_{s_p}^{i_p}\dots\mathrm{d}W_{s_2}^{i_2},$$

which by the induction assumption satisfies $\mathbb{E}(I_n^2(i_p,\dots,i_2)[s_1])=\mathcal{O}((s_1-t_n)^{2q^*})=\mathcal{O}((\Delta t)^{2q^*})$, where

$$q^*=\sum_{j=2}^{p}\frac{2-\min\{1,i_j\}}{2}.$$

There are two cases: First, if $i_1=0$, by the Cauchy–Schwarz inequality and the induction hypothesis we have

$$\begin{aligned}\mathbb{E}(I_n^2(i_p,\dots,i_1))&=\mathbb{E}\Big(\int_{t_n}^{t_{n+1}}I_n(i_p,\dots,i_2)[s_1]\mathrm{d}s_1\Big)^2\\&\leqslant\int_{t_n}^{t_{n+1}}\mathbb{E}\big[I_n(i_p,\dots,i_2)[s_1]\big]^2\mathrm{d}s_1\Delta t\\&\leqslant\int_{t_n}^{t_{n+1}}\mathbb{E}\big[I_n(i_p,\dots,i_2)[t_{n+1}]\big]^2\mathrm{d}s_1\Delta t\\&\leqslant C(\Delta t)^{2q^*}(\Delta t)^2.\end{aligned}\tag{2.1}$$

Second, if $i_1\in\{1,\dots m\}$, we use the Itô isometry

$$\begin{aligned}\mathbb{E}(I_n^2(i_p,\dots,i_1))&=\mathbb{E}\Big(\Big(\int_{t_n}^{t_{n+1}}I_n(i_p,\dots,i_2)[s_1]\mathrm{d}W_{i_1}(s_1)\Big)^2\Big)=\int_{t_n}^{t_{n+1}}\mathbb{E}I_n^2(i_p,\dots,i_2)[s_1]\mathrm{d}s_1\\&\leqslant C(\Delta t)^{2q^*}\int_{t_n}^{t_{n+1}}\mathrm{d}s_1=C(\Delta t)^{2q^*+1},\end{aligned}$$

which verifies the claim. Therefore $\mathbb{E}(I_n^2(i_p,\dots,i_1))=\mathcal{O}((\Delta t)^{2q})$, where

$$\begin{aligned}2q&=\begin{cases}2q^*+2 & \text{if } i_1=0,\\ 2q^*+1 & \text{if } i_1\in\{1,\dots,m\}\end{cases}\\&=2q^*+2-\min\{1,i_1\}\\&=\sum_{j=1}^{p}(2-\min\{1,i_j\}).\end{aligned}$$

Therefore, the induction step is completed and the result is proved.
Last point follows easily by applying Cauchy-Schwarz inequality and using point *iii*).

Exercise 3.

The aim of this exercise is to derive a method of strong order 3/2 for an autonomous one-dimensional SDE of the form

$$\mathrm{d}X(t)=f(X(t))\mathrm{d}t+g(X(t))\mathrm{d}W(t).$$

Let us first define the following Itô integrals with the notation of Exercise 3

$$
\begin{aligned}
I_n(0) &= \int_{t_n}^{t_{n+1}} \mathrm{d}s_1, & I_n(1) &= \int_{t_n}^{t_{n+1}} \mathrm{d}W(s_1),\\
I_n(0,0) &= \int_{t_n}^{t_{n+1}} \int_{t_n}^{s_1} \mathrm{d}s_2 \mathrm{d}s_1, & I_n(0,1) &= \int_{t_n}^{t_{n+1}} \int_{t_n}^{s_1} \mathrm{d}W_{s_2} \mathrm{d}s_1,\\
I_n(1,0) &= \int_{t_n}^{t_{n+1}} \int_{t_n}^{s_1} \mathrm{d}s_2 \mathrm{d}W_{s_1}, & I_n(1,1) &= \int_{t_n}^{t_{n+1}} \int_{t_n}^{s_1} \mathrm{d}W_{s_2} \mathrm{d}W_{s_1},\\
I_n(1,1,1) &= \int_{t_n}^{t_{n+1}} \int_{t_n}^{s_1} \int_{t_n}^{s_2} \mathrm{d}W_{s_3} \mathrm{d}W_{s_2} \mathrm{d}W_{s_1}, & \dots
\end{aligned}
$$

In the following, we denote by f, f', f'' and g, g', g'' the values $f(X(t_n)), f'(X(t_n)), f''(X(t_n))$ and $g(X(t_n)), g'(X(t_n)), g''(X(t_n)$ respectively. Moreover, we define

$$
\begin{aligned}
e_1 &= \int_{t_n}^{t_{n+1}} f(X(s_1))\mathrm{d}s_1 - f\Delta t,\\
e_2 &= \int_{t_n}^{t_{n+1}} g(X(s_1))\mathrm{d}W_{s_1} - g\Delta W_n.
\end{aligned}
$$

i) Show that

$$
\begin{aligned}
e_1 = {} & f'gI_n(0,1) + (ff' + \frac{1}{2}g^2 f'')I_n(0,0)\\
& + \mathcal{O}(I_n(0,1,0)) + \mathcal{O}(I_n(0,1,1)) + \mathcal{O}(I_n(0,0,0)) + \mathcal{O}(I_n(0,0,1)).
\end{aligned}
$$

ii) Show that

$$
\begin{aligned}
e_2 = {} & (fg' + \frac{1}{2}g^2 g'')I_n(1,0) + gg'I_n(1,1) + g(gg'' + (g')^2)I_n(1,1,1)\\
& + \mathcal{O}(I_n(1,0,0)) + \mathcal{O}(I_n(1,0,1)) + \mathcal{O}(I_n(1,1,0)) + \mathcal{O}(I_n(1,1,1,0)) + \mathcal{O}(I_n(1,1,1,1)).
\end{aligned}
$$

Then, a possible proposal for a method of strong order 3/2 is given by

$$
\begin{aligned}
Y_{n+1} = {} & Y_n + f\Delta t + g\Delta W_n + f'gI_n(0,1) + \Big(ff' + \frac{1}{2}g^2 f''\Big)I_n(0,0)\\
& + \Big(fg' + \frac{1}{2}g^2 g''\Big)I_n(1,0) + gg'I_n(1,1) + g(gg'' + (g')^2)I_n(1,1,1).
\end{aligned}
\tag{3.1}
$$

iii) Using the results of Exercise 1, write explicitly the method (3.1).

Solution

i) Applying Itô formula to $f(X(s_1))$ we get

$$
\begin{aligned}
e_1 = {} & \int_{t_n}^{t_{n+1}} \int_{t_n}^{s_1} (f'(X(s_2))f(X(s_2)) + \frac{1}{2}f''(X(s_2))g^2(X(s_2)))\mathrm{d}s_2\mathrm{d}s_1\\
& + \int_{t_n}^{t_{n+1}} \int_{t_n}^{s_1} f'(X(s_2))g(X(s_2))\mathrm{d}W_{s_2}\mathrm{d}s_1.
\end{aligned}
\tag{3.2}
$$

Applying Itô formula again to the integrands and keeping only the terms in $s_2 = t_n$ explicitly, we have

$$
\begin{aligned}
e_1 = {} & \int_{t_n}^{t_{n+1}} \int_{t_n}^{s_1} (ff' + \frac{1}{2}g^2 f'')\mathrm{d}s_2\mathrm{d}s_1 + \int_{t_n}^{t_{n+1}} \int_{t_n}^{s_1} f'g\mathrm{d}W_{s_2}\mathrm{d}s_1\\
& + \mathcal{O}(I_n(0,1,0)) + \mathcal{O}(I_n(0,1,1)) + \mathcal{O}(I_n(0,0,0)) + \mathcal{O}(I_n(0,0,1)).
\end{aligned}
$$

which is the desired result. Notice that triple integrals are all of order $(\Delta t)^2$ or higher.

ii) Applying Itô formula to $g(X(s_1))$ we get

$$\begin{aligned} e_2 &= \int_{t_n}^{t_{n+1}}\int_{t_n}^{s_1}(g'(X(s_2))f(X(s_2)) + \frac{1}{2}g''(X(s_2))g^2(X(s_2)))\mathrm{d}s_2\mathrm{d}W_{s_1} \\ &\quad + \int_{t_n}^{t_{n+1}}\int_{t_n}^{s_1} g'(X(s_2))g(X(s_2))\mathrm{d}W_{s_2}\mathrm{d}W_{s_1}. \end{aligned} \tag{3.3}$$

Applying Itô formula again to the integrands we obtain

$$\begin{aligned} e_2 &= \int_{t_n}^{t_{n+1}}\int_{t_n}^{s_1}(fg' + \frac{1}{2}g^2g'')\mathrm{d}s_2\mathrm{d}W_{s_1} + \int_{t_n}^{t_{n+1}}\int_{t_n}^{s_1} g'g\mathrm{d}W_{s_2}\mathrm{d}W_{s_1} \\ &\quad + \int_{t_n}^{t_{n+1}}\int_{t_n}^{s_1}\int_{t_n}^{s_2} g(X(s_3))(g(X(s_3))g''(X(s_3)) + g'(X(s_3))^2)\mathrm{d}W_{s_3}\mathrm{d}W_{s_2}\mathrm{d}W_{s_1} \\ &\quad + \mathcal{O}(I_n(1,1,0)) + \mathcal{O}(I_n(1,0,1)) + \mathcal{O}(I_n(1,0,0)), \end{aligned}$$

where the integrand in the triple Itô integral is of order $(\Delta t)^{3/2}$ and is kept explicitly. Applying Itô formula once more to this term, we get

$$\begin{aligned} e_2 &= \int_{t_n}^{t_{n+1}}\int_{t_n}^{s_1}(fg' + \frac{1}{2}g^2g'')\mathrm{d}s_2\mathrm{d}W_{s_1} + \int_{t_n}^{t_{n+1}}\int_{t_n}^{s_1} g'g\mathrm{d}W_{s_2}\mathrm{d}W_{s_1} \\ &\quad + \int_{t_n}^{t_{n+1}}\int_{t_n}^{s_1}\int_{0}^{s_2} g(gg'' + (g')^2)\mathrm{d}W_{s_3}\mathrm{d}W_{s_2}\mathrm{d}W_{s_1} \\ &\quad + \mathcal{O}(I_n(1,1,0)) + \mathcal{O}(I_n(1,0,1)) + \mathcal{O}(I_n(1,0,0)) + \mathcal{O}(I_n(1,1,1,0)) + \mathcal{O}(I_n(1,1,1,1)), \end{aligned}$$

which is the desired result. Notice that all the triple and quadruple integrals are of order $(\Delta t)^2$ or higher, except of the term in $I_n(1,1,1)$ which is kept in the scheme (3.1) and is of order $(\Delta t)^{3/2}$.

iii) The method reads

$$\begin{aligned} Y_{n+1} &= Y_n + f\Delta t + g\Delta W + \frac{1}{2}gg'((\Delta W)^2 - \Delta t) \\ &\quad + f'g\Delta Z + \frac{1}{2}\Big(ff' + \frac{1}{2}g^2f''\Big)(\Delta t)^2 \\ &\quad + \Big(fg' + \frac{1}{2}g^2g''\Big)(\Delta t\Delta W - \Delta Z) \\ &\quad + \frac{1}{2}g(gg'' + (g')^2)\Big(\frac{1}{3}(\Delta W)^2 - \Delta t\Big)\Delta W. \end{aligned}$$

Exercise 4.

Let $a \in \mathbb{R}$ and consider the SDE on $[0,T]$

$$\mathrm{d}X(t) = -\frac{1}{2}a^2X(t) + a\sqrt{1 - X(t)^2}\mathrm{d}W(t), \tag{4.1}$$

with initial condition $X(0) = X_0$.

i) Verify that the Itô solution of equation (4.1) is given by

$$X(t) = \sin(aW(t) + \arcsin(X_0)).$$

ii) Write the Milstein scheme for equation (4.1) and the corresponding derivative-free schemes with supporting values

$$\begin{aligned} \widehat{Z}_n &= X_n + f(t_n, X_n)\Delta t + g(t_n, X_n)\sqrt{h}, \\ \widetilde{Z}_n &= X_n + g(t_n, X_n)\sqrt{\Delta t}. \end{aligned} \tag{4.2}$$

iii) Implement the schemes above and verify that they all have strong order of convergence 1. Set $a = 0.5$, $X_0 = 0$, $T = 1$. Choose different step sizes $\Delta t = 2^{-i}$ with $i = 2, \dots, 8$ and approximate the expectations using $M = 10^4$ realizations of the Brownian motion. What difference do you notice between the three schemes?

iv) Repeat the previous point for the strong order 1.5 scheme of Exercise 4.

Solution

i) We apply the Itô's formula to $X(t)$.

ii) The Milstein scheme is given by

$$\begin{aligned} X_{n+1} &= X_n - \frac{1}{2}a^2 X_n \Delta t + a\sqrt{1-X_n^2}\Delta W_n - \frac{1}{2}a^2\sqrt{1-X_n^2}\frac{X_n}{\sqrt{1-X_n^2}}((\Delta W_n)^2 - \Delta t) \\ &= X_n - \frac{1}{2}a^2 X_n \Delta t + a\sqrt{1-X_n^2}\Delta W_n - \frac{1}{2}a^2 X_n((\Delta W_n)^2 - \Delta t) \\ &= X_n + a\sqrt{1-X_n^2}\Delta W_n - \frac{1}{2}a^2 X_n(\Delta W_n)^2. \end{aligned} \tag{4.3}$$

The supporting values are

$$\begin{aligned} \widehat{Z}_n &= X_n - \frac{1}{2}a^2 X_n \Delta t + a\sqrt{1-X_n^2}\sqrt{\Delta t}, \\ \widetilde{Z}_n &= X_n + a\sqrt{1-X_n^2}\sqrt{\Delta t}. \end{aligned} \tag{4.4}$$

We then replace them in the definition of the derivative-free Milstein scheme

$$X_{n+1} = X_n - \frac{1}{2}a^2 X_n \Delta t + a\sqrt{1-X_n^2}\Delta W_n + \frac{a}{\sqrt{\Delta t}}\left(\sqrt{1-Z_n^2} - \sqrt{1-X_n^2}\right)((\Delta W_n)^2 - \Delta t). \tag{4.5}$$

iii) The plot given in Figure 1 shows that the convergence rate is maintained in both the derivative free versions. Nonetheless, the constant is slightly smaller for the original version. Errors with the supporting value $\widehat{Z}$ and $\tilde{Z}$ are approximately the same, even though for $\widehat{Z}$ they are slightly lower.

iv) Considering the previous exercise, we have the following values

$$f'(x) = -\frac{1}{2}a^2,\ f''(x) = 0,\ g'(x) = -\frac{ax}{\sqrt{1-x^2}},\ g''(x) = -\frac{1}{(1-x^2)^{3/2}}.$$

This implies that

$$f'g = -\frac{1}{2}a^2 a\sqrt{1-x^2} = -\frac{1}{2}a^3\sqrt{1-x^2}, \quad \left(ff' + \frac{1}{2}g^2 f''\right) = \frac{1}{4}a^4 x$$

$$\begin{aligned} \left(fg' + \frac{1}{2}g^2 g''\right) &= \frac{a^3x^2}{2\sqrt{1-x^2}} + \frac{1}{2}a^2(1-x^2)\cdot -\frac{a}{(1-x^2)^{3/2}} = \\ &= \frac{1}{2}\frac{a^3x^2}{\sqrt{1-x^2}} - \frac{1}{2}\frac{a^3}{\sqrt{1-x^2}} \\ &= -\frac{1}{2}\frac{a^3(1-x^2)}{\sqrt{1-x^2}} = -\frac{1}{2}a^3\sqrt{1-x^2}, \end{aligned}$$

$$\begin{aligned} \frac{1}{2}g(gg'' + (g')^2) &= \frac{1}{2}a\sqrt{1-x^2}\left(a\sqrt{1-x^2}\cdot -\frac{a}{(1-x^2)^{3/2}} + (-\frac{ax}{\sqrt{1-x^2}})^2\right) \\ &= \frac{1}{2}\left(-a^3\frac{1}{\sqrt{1-x^2}} + \frac{a^3x^2}{\sqrt{1-x^2}}\right) \\ &= -\frac{1}{2}a^3\sqrt{1-x^2}. \end{aligned}$$

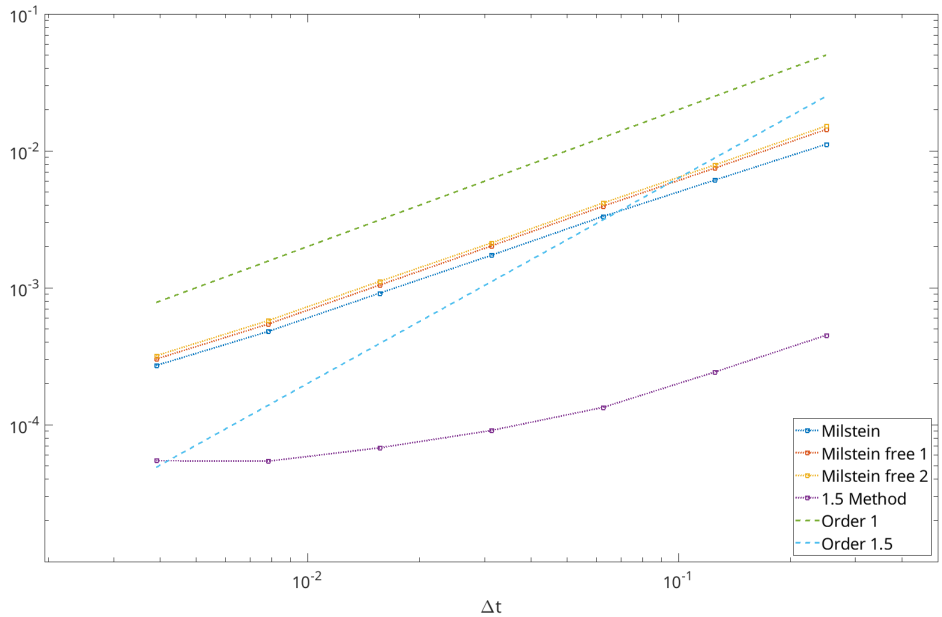

Figure 1: Order of convergence of the Milstein scheme, its derivative-free schemes and the method of Exercise 3.

Therefore, our method will turn out to be

$$
\begin{aligned}
X_{n+1} = {} & X_n + f\Delta t + g\Delta W + \frac{1}{2}gg'((\Delta W)^2 - \Delta t) \\
& + f'g\Delta Z + \frac{1}{2}\Big(ff' + \frac{1}{2}g^2 f''\Big)(\Delta t)^2 \\
& + \Big(fg' + \frac{1}{2}g^2 g''\Big)(\Delta t\Delta W - \Delta Z) \\
& + \frac{1}{2}g(gg'' + (g')^2)\Big(\frac{1}{3}(\Delta W)^2 - \Delta t\Big)\Delta W. \\
= {} & X_n + f\Delta t + g\Delta W + \frac{1}{2}gg'((\Delta W)^2 - \Delta t) + \frac{1}{8}a^4 X_n(\Delta t)^2 \\
& + \Big(-\frac{1}{2}a^3\sqrt{1 - X_n^2}\Big)\Big(\Delta Z + \Delta t\Delta W - \Delta Z + \Big(\frac{1}{3}(\Delta W)^2 - \Delta t\Big)\Delta W\Big) \\
= {} & X_n - \frac{1}{2}a^2 X_n\Delta t + a\sqrt{1 - X_n^2}\Delta W - \frac{1}{2}a^2 X_n((\Delta W)^2 - \Delta t) + \frac{1}{8}a^4 X_n(\Delta t)^2 \\
& + \Big(-\frac{1}{2}a^3\sqrt{1 - X_n^2}\Big)\Big(\Delta Z + \Delta t\Delta W - \Delta Z + \Big(\frac{1}{3}(\Delta W)^2 - \Delta t\Big)\Delta W\Big) \\
= {} & X_n + a\sqrt{1 - X_n^2}\Delta W - \frac{1}{2}a^2 X_n(\Delta W)^2 + \frac{1}{8}a^4 X_n(\Delta t)^2 + \Big(-\frac{1}{2}a^3\sqrt{1 - X_n^2}\Big)\Big(\frac{1}{3}(\Delta W)^3\Big).
\end{aligned}
$$

Exercise 5.

Consider the SDE on $[0, 1]$

$$\mathrm{d}X(t) = -\frac{1}{2}X(t)\mathrm{d}t + X(t)\mathrm{d}W_1(t) + X(t)\mathrm{d}W_2(t),$$

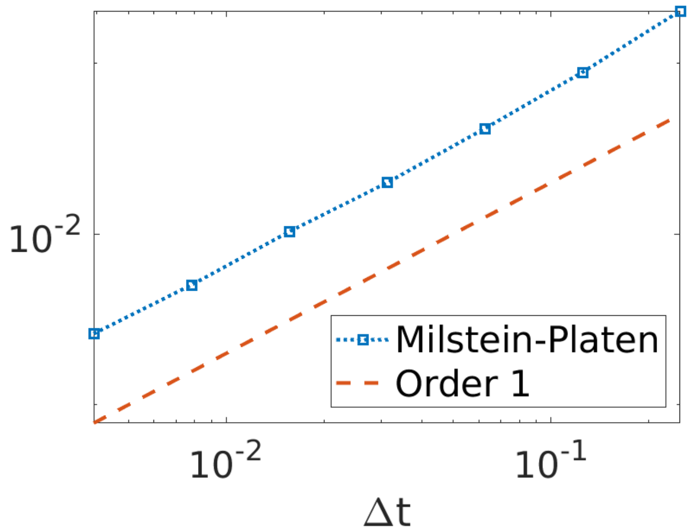

Figure 2: Order of convergence of the Milstein–Platen scheme of Exercise 7.

with initial condition $X(0) = 1$. Does this problem have commutative noise? Give the exact solution and compute the numerical solution employing the Milstein–Platen scheme. Verify that it has strong order of convergence 1. Choose different step sizes $\Delta t = 2^{-i}$ with $i = 2, \dots, 8$ and approximate the expectations using $M = 10^4$ realizations of the Brownian motion.

Solution

The noise is linear and therefore commutative. In order to compute the exact solution we define

$$\widetilde{W}(t) = \frac{1}{\sqrt{2}}(W_1(t) + W_2(t)),$$

which is a Brownian motion. Hence, we can rewrite the equation as

$$\mathrm{d}X(t) = -\frac{1}{2}X(t)\mathrm{d}t + \sqrt{2}X(t)\mathrm{d}\widetilde{W}(t),$$

whose exact solution is

$$X(t) = e^{-\frac{3}{2}t+\sqrt{2}\widetilde{W}(t)} = e^{-\frac{3}{2}t+W_1(t)+W_2(t)}.$$

The Milstein–Platen scheme for this problem is given by

$$X_{n+1} = X_n\left[1 - \frac{1}{2}h + I^n_{(1)} + I^n_{(2)} + I^n_{(1,1)} + I^n_{(2,2)} + (I^n_{(1,2)} + I^n_{(2,1)})\right],$$

where $I^n_{(1)} = \Delta_n W_1$, $I^n_{(2)} = \Delta_n W_2$, $I^n_{(1,1)} = ((\Delta_n W_1)^2 - h)/2$, $I^n_{(2,2)} = ((\Delta_n W_2)^2 - h)/2$ and $I^n_{(1,2)} + I^n_{(2,1)} = \Delta_n W_1 \Delta_n W_2$. The plot is then given in Figure 2 and shows that the strong order of convergence is 1.