Series 7 - November 6, 2024

Exercise 1.

Consider the following SDE

$$\begin{aligned} \mathrm{d}X_t &= \mu X_t \mathrm{d}t + \sigma X_t \mathrm{d}W(t), \quad t \in [0,T] \\ X(0) &= X_0, \end{aligned} \tag{1.1}$$

with $X(0) \in \mathbb{R}$, $\mu \in \mathbb{R}$, $\sigma > 0$. Equation (1.1) admits a unique closed-form solution and it is called Geometric Brownian motion.

1) Find the closed form solution for (1.1).
2) Consider $X(0) = 1$, $\mu = 2$, $\sigma = 1$, and $T = 1$. Compute the Euler-Maruyama discretization $\{X_n\}_{n=0}^N$ of (1.1) for $\Delta t = \frac{T}{n} = 0.1, 0.05, 0.01, 0.005, 0.001$ and for a single realization plot $\sup_{0\leqslant n\leqslant N} |X_{t_n} - X_n|$ versus $(\Delta t)^{\frac{1}{2}}$.
3) Compute $Z_N = \sup_{0\leqslant n\leqslant N} \frac{|X_{t_n} - X_n|}{(\Delta t)^{\frac{1}{2}}}$ for $\Delta t = T/N$ with $N = 100$ and $M = 1000$ independent trajectories and plot the empirical $\mathrm{CDF}(Z_N)$. Do the same computation for $N = 1000$; does $\mathrm{CDF}(Z_N)$ converge to a limit distribution ?
4) What happens if you take $Z_N^\alpha = \sup_{0\leqslant n\leqslant N} \frac{|X_{t_n} - X_n|}{(\Delta t)^\alpha}$ for $\alpha < \frac{1}{2}$? Is this result consistent with your expectations ?

Solution

1) Formally, the equation gives
$$\frac{\mathrm{d}X}{X} = \mu \mathrm{d}t + \sigma \mathrm{d}W.$$
Hence, applying the Itô formula to $u(x) = \log(x)$, we obtain
$$\mathrm{d}u(X) = \frac{\mathrm{d}X}{X} - \frac{1}{2X^2}\sigma^2 X^2 \mathrm{d}t = (\mu - \frac{1}{2}\sigma^2)\mathrm{d}t + g\mathrm{d}W.$$
Therefore, we have
$$\log(X(t)) = \log(X_0) + \int_0^t \Big(\mu - \frac{1}{2}\sigma^2\Big)\mathrm{d}s + \int_0^t \sigma \mathrm{d}W(s),$$
which implies
$$X(t) = X_0 e^{\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma W(t)}.$$
Finally, we verify rigorously that $X(t)$ is a strong solution of the SDE, indeed we know that:
i) $t \mapsto X(t)$ is continuous,
ii) $X(t)$ is $\mathcal{F}(t)$-adapted,
iii) $(t,\omega) \mapsto \mu X(t,\omega)$ is in $\mathrm{M}^1(0,T)$ and $(t,\omega) \mapsto \sigma X(t,\omega)$ is in $\mathrm{M}^2(0,T)$,
iv) the equation holds a.s. for each t.

Since f and g are continuous then *iii)* is verified because the mappings are progressively measurable.

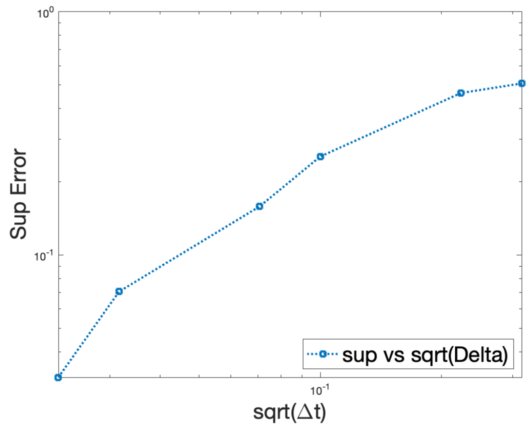

Figure 1: Sup Error computed over time with respect to one trajectory vd $\sqrt{\Delta t}$

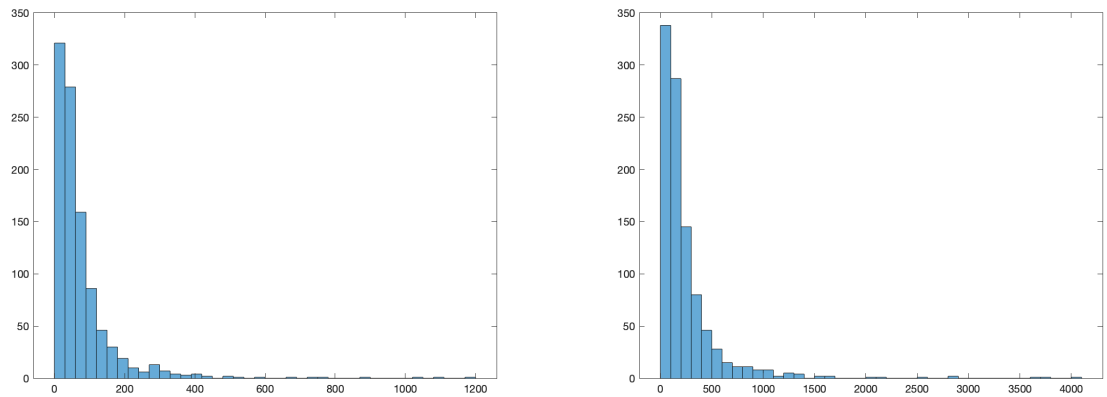

Figure 2: Histogram of the error for $M = 1000$ trajectories for $N = 100$ and $N = 1000$.

2)

3)

Exercise 2.

Consider the $1D$ autonomous SDE

$$\begin{cases} dX_t &= b(X_t)dt + \sigma(X_t)dW_t, \quad t \in [0,T], \\ X(0) &= \eta \in L^2(\Omega), \end{cases} \tag{2.1}$$

and the stochastic θ-method

$$X_{n+1} = X_n + \theta \Delta t b(X_{n+1}) + (1-\theta)\Delta t b(X_n) + \sigma(X_n)\Delta W_n \tag{2.2}$$

Assume b, σ to satisfy a global Lipschitz condition and a linear growth bound and, moreover, b is continuously differentiable with bounded derivative $|b'(x)| \leqslant K$ for all $x \in \mathbb{R}$.

1) Show that if $\Delta t < \frac{1}{K\theta}$ the numerical solution is well defined (uniquely exists)

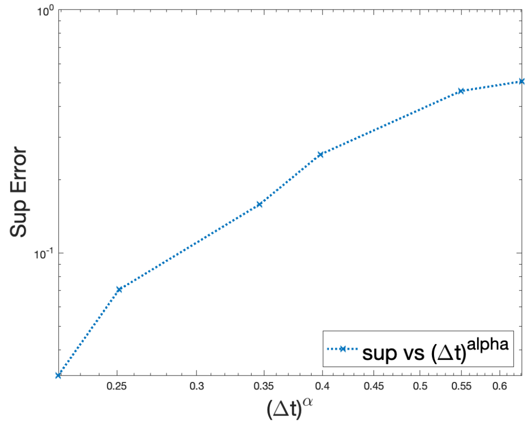

Figure 3: Sup Error computed over time with respect to one trajectory vs $(\varDelta t)^\alpha$, $\alpha = 0.2$.

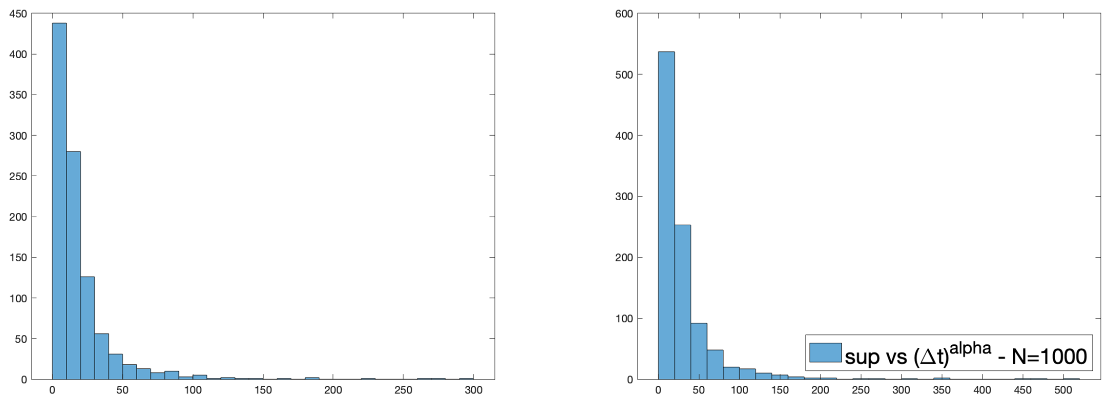

Figure 4: Histogram of the error for $M = 1000$ trajectories for $N = 100$ (left) and $N = 1000$ (right).

Hint. You can show that the map $\varphi(x) = \theta b(x)\varDelta t + \gamma$, $\gamma = X_n + (1-\theta)b(X_n)\varDelta t + \sigma(X_n)\varDelta W_n$ is contractive

2) Show that $\exists C > 0 \;:\; \sup\limits_{0\leqslant n\leqslant N} \mathbb{E}[|X_n|^2] \leqslant C$

Hint. multiply (2.2) by X_n and use the identity $(X_{n+1} - X_n)X_n = \frac{1}{2}X_{n+1}^2 - \frac{1}{2}X_n^2 - \frac{1}{2}(X_{n+1} - X_n)^2$

3) Defining the (non adapted) process

$$\begin{aligned}\hat{X}_s &= X_n + (\theta b(X_{n+1}) + (1-\theta)b(X_n))(s - t_n) + \sigma(X_n)(W_s - W_{t_n}) \\ &= X_n + \int_{t_n}^{s} (\theta b(X_{n+1}) + (1-\theta)b(X_n))d\tau + \int_{t_n}^{s} \sigma(X_n)dW_\tau, \quad t_n < s \leqslant t_{n+1}.\end{aligned} \tag{2.3}$$

Redo the same steps of the proof of the strong convergence of Euler-Maruyama to show that $\mathbb{E}[\sup\limits_{0\leqslant t\leqslant T} |X_s - \hat{X}_s|^2] \leqslant C\varDelta t$.

Solution

1) Considering $\varphi(x) = \theta b(x)\Delta t + \gamma$, $\gamma = X_n + (1-\theta)b(X_n)\Delta t + \sigma(X_n)\Delta W_n$, we have

$$\begin{aligned}|\varphi(x)-\varphi(y)| &\leqslant \theta\Delta t|b(x)-b(y)|\\ &\leqslant K\theta\Delta t|x-y|\\ &< |x-y|\end{aligned} \tag{2.4}$$

where in the last line we use the condition $\Delta t < \frac{1}{K\theta}$. Therefore, under this condition, φ is a contractive map and, via the Banach theorem, there exists a unique fixed point y of φ, i.e. $y = \varphi(y)$. Setting $X_{n+1} = y$ we are done.

2) We have

$$\begin{aligned}\frac{1}{2}X_{n+1}^2 - \frac{1}{2}X_n^2 - \frac{1}{2}(X_{n+1}-X_n)^2 =&(X_{n+1}-X_n)X_n\\ =&\theta\Delta t b(X_{n+1})X_n + (1-\theta)\Delta t b(X_n)X_n + \sigma(X_n)\Delta W_n X_n\\ \leqslant&\theta\Delta t K|X_{n+1}-X_n||X_n| + \Delta t b(X_n)X_n + \sigma(X_n)\Delta W_n X_n\\ \implies X_{n+1}^2 \leqslant& X_n^2 + (X_{n+1}-X_n)^2 + \Delta t\theta K|X_{n+1}-X_n|^2 + \Delta t\theta K|X_n|^2\\ &+ 2\Delta t b(X_n)X_n + 2\sigma(X_n)\Delta W_n X_n\end{aligned} \tag{2.5}$$

Moreover,

$$|X_{n+1}-X_n| \leqslant \theta\Delta t|b(X_{n+1})-b(X_n)| + \Delta t|b(X_n)| + |\sigma(X_n)\Delta W_n| \tag{2.6}$$

and, therefore, using the Lipschitz continuity

$$\begin{aligned}|X_{n+1}-X_n| &\leqslant \frac{1}{1-\theta\Delta K}(\Delta t|b(X_n)| + |\sigma(X_n)\Delta W_n|)\\ \implies |X_{n+1}-X_n|^2 &\leqslant \frac{2}{(1-\theta\Delta tK)^2}(\Delta t)^2|b(X_n)|^2 + \frac{2}{(1-\theta\Delta tK)^2}|\sigma(X_n)\Delta W_n|^2\end{aligned} \tag{2.7}$$

Taking expectation we get

$$\begin{aligned}\mathbb{E}[X_{n+1}^2] \leqslant& \mathbb{E}[X_n^2] + 2\mathbb{E}[b(X_n)^2]\frac{2(1+\theta\Delta tK)}{(1-\theta t\Delta K)^2}(\Delta t)^2 + \frac{2(1+\theta\Delta tK)}{(1-\theta\Delta tK)^2}\mathbb{E}[(\sigma(X_n)\Delta W_n)^2]\\ &+ \theta\Delta tK\mathbb{E}[X_n^2] + 2\Delta t\mathbb{E}[X_n b(X_n)]\\ &\leqslant \mathbb{E}[X_n^2] + 2\mathbb{E}[X_n^2]\frac{2(1+\theta\Delta tK)((\Delta t)^2+\Delta t)}{(1-\theta t\Delta K)^2} + \frac{2(1+\theta\Delta tK)((\Delta t)^2+\Delta t)}{(1-\theta t\Delta K)^2}K\\ &+ \theta\Delta tK\mathbb{E}[X_n^2] + 2\Delta t\mathbb{E}[X_n^2] + 2\Delta tK(1+\mathbb{E}[X_n^2])\\ &= \mathbb{E}[X_n^2](1 + \left(\frac{2(1+\theta\Delta tK)((\Delta t)+1)}{(1-\theta t\Delta K)^2} + \theta + 2 + 2K\right)\Delta t\\ &+ 2K\left(\frac{2(1+\theta\Delta tK)((\Delta t)+1)}{(1-\theta t\Delta K)^2} + 2K\right)\Delta t\end{aligned} \tag{2.8}$$

where in the second line we use the linear-growth bound and the Young's inequality. For all $\epsilon > 0$ small enough, having $\Delta t < \frac{1}{K\theta} - \epsilon$ guarantees to find explicit positive constant $C, D > 0$ such that

$$\mathbb{E}[X_{n+1}^2] \leqslant \mathbb{E}[X_n^2](1 + C\Delta t) + D\Delta t. \tag{2.9}$$

Now, we prove by induction that for all n it holds:

$$\mathbb{E}[X_n^2] \leqslant (\mathbb{E}[X_0^2] + Dn\Delta t)\exp\{Cn\Delta t\}. \tag{2.10}$$

Obviously, (2.10) holds for $n = 0$. Now, suppose that (2.10) is valid for n, then from (2.9)

$$\begin{aligned}\mathbb{E}[X_{n+1}^2] &\leqslant \mathbb{E}[X_n^2](1 + C\Delta t) + D\Delta t\\ &\leqslant (\mathbb{E}[X_0^2] + Dn\Delta t)\exp\{Cn\Delta t\}(1 + C\Delta t) + D\Delta t\\ &\leqslant (\mathbb{E}[X_0^2] + Dn\Delta t)\exp\{Cn\Delta t\}\exp\{C\Delta t\} + (\mathbb{E}[X_0^2] + Dn\Delta t)\exp\{Cn\Delta t\}\exp\{C\Delta t\}\\ &\leqslant (\mathbb{E}[X_0^2] + D(n+1)\Delta t)\exp\{C(n+1)\Delta t\}.\end{aligned} \tag{2.11}$$

Then, for all n one has

$$\mathbb{E}[X_n^2] \leqslant (\mathbb{E}[X_0^2] + DT)\exp\{CT\}. \tag{2.12}$$

Alternately to induction, one could also have explored some discrete Gronwall lemma inequality.

3) We defined the inteporlated process starting from the numerical one as done in the Euler-Maruyama convergence proof.

$$\begin{aligned} X_{n+1} =& X_n + \theta \Delta t b(X_{n+1}) + (1-\theta)\Delta t b(X_n) + \sigma(X_n)\Delta W_n \\ \hat{X}_t =& X_n + \theta b(X_{n+1})(t-t_n) + (1-\theta) b(X_n)(t-t_n), \quad t_n < t \leqslant t_{n+1} \\ &+ \sigma(X_n)\big(W_t - W_{t_n}\big), \quad t = t_n + t_1 \end{aligned} \tag{2.13}$$

Moreover, let us denote $n_s = \max\{n \, : \, t_n < s\}$ and $\varphi_s = t_{n_s}$. Then one has the following splitting $X_t - \hat{X}_t$

$$\begin{aligned} X_t - \hat{X}_t &= \int_0^t b(X_s) - \big[(1-\theta)b\big(X_{n_s+1}\big) + \theta b\big(X_{n_s}\big)\big] ds + \int_0^t \sigma(X_s) - \sigma\big(X_{n_s}\big) dW_s \\ &= \int_0^t b(X_s) - b\big(X_{\varphi_s}\big) ds + \int_0^t \sigma(X_s) - \sigma\big(X_{\varphi_s}\big) dW_s \\ &+ \int_0^t b\big(X_{\varphi_s}\big) - \big[(1-\theta)b\big(X_{n_s+1}\big) + \theta b\big(X_{n_s}\big)\big] ds + \int_0^t \sigma\big(X_{\varphi_s}\big) - \sigma\big(X_{n_s}\big) dW_s \end{aligned} \tag{2.14}$$

Passing to the sup and apply the average we get

$$\begin{aligned} \mathbb{E}\left[\sup_{0\leqslant t\leqslant T}\big|X_t - \hat{X}_t\big|^2\right] \leqslant = 4\mathbb{E}\left[\sup_{0\leqslant t\leqslant T}\left|\int_0^t b(X_s) - b\big(X_{\varphi_s}\big) ds\right|^2\right] + 4\mathbb{E}\left[\sup_{0\leqslant t\leqslant T}\left|\int_0^t \sigma(X_s) - \sigma\big(X_{\varphi_s}\big) dW_s\right|^2\right] \\ + 4\mathbb{E}\left[\sup_{0\leqslant t\leqslant T}\left|\int_0^t b\big(X_{\varphi_s}\big) - \big[(1-\theta)b\big(X_{n_s+1}\big) + \theta b\big(X_{n_s}\big)\big] ds\right|^2\right] \\ + 4\mathbb{E}\left[\sup_{0\leqslant t\leqslant T}\left|\int_0^t \sigma\big(X_{\varphi_s}\big) - \sigma\big(X_{n_s}\big) dW_s\right|^2\right] \end{aligned} \tag{2.15}$$

The elements that appears also in the Euler-Maruyama proof are treated in the same way. Instead, for the only different one we have thanks to the standard assumption and point 2) that

$$\begin{aligned} &\mathbb{E}\left[\sup_{0\leqslant t\leqslant T}\left|\int_0^t b\big(X_{\varphi_s}\big) - \big[(1-\theta)b\big(X_{n_s+1}\big) + \theta b\big(X_{n_s}\big)\big] ds\right|^2\right] \\ &\leqslant \mathbb{E}\left[\sup_{0\leqslant t\leqslant T}\left|\int_0^t b\big(X_{\varphi_s}\big) - b\big(X_{n_s}\big) + \theta b\big(X_{n_s}\big) - b\big(X_{n_s+1}\big) ds\right|^2\right] \\ &\leqslant \mathbb{E}\left[\sup_{0\leqslant t\leqslant T} T\int_0^t 2\big|b\big(X_{\varphi_s}\big) - b\big(X_{n_s}\big)\big|^2 + \theta 2\big|b\big(X_{n_s}\big) - b\big(X_{n_s+1}\big)\big|^2 ds\right] \\ &\leqslant \mathbb{E}\left[\sup_{0\leqslant t\leqslant T} T\int_0^t 2\big|b\big(X_{\varphi_s}\big) - b\big(X_{n_s}\big)\big|^2 + \theta 2K\big|X_{n_s} - X_{n_s+1}\big|^2 ds\right] \\ &= T\int_0^T \mathbb{E}\left[\sup_{0\leqslant s\leqslant r} 2\big|b\big(X_{\varphi_s}\big) - b\big(X_{n_s}\big)\big|^2 + \theta 2K\big|\theta\Delta t b\big(X_{n_s}\big) + (1-\theta)\Delta t b\big(X_{n_s+1}\big) + \sigma\big(X_{n_s}\big)\Delta W_n\big|^2\right] dr \\ &\leqslant T\int_0^T 2K\mathbb{E}\left[\sup_{0\leqslant s\leqslant r}\big|X_{\varphi_s} - X_{n_s}\big|^2 + \theta 2K^2(1+C)((\Delta t)^2 + \Delta t)\right] ds \end{aligned} \tag{2.16}$$

and then conclusion follows similarly to the Euler-Maruyama proof of convergence.

Exercise 3.

Let $b > 0$, $\sigma \in \mathbb{R}$ and $X_0 \in \mathbb{R}$ and consider the Langevin equation

$$\begin{aligned} \mathrm{d}X(t) &= -bX(t)\mathrm{d}t + \sigma \mathrm{d}W(t), \quad t \in [0,T], \\ X(0) &= X_0. \end{aligned} \tag{3.1}$$

Remark. The solution $X(t)$ is also called the Ornstein–Uhlenbeck process.

i) Solve equation (3.1).

ii) Verify that $\lim_{t\to\infty} \mathbb{E}[X(t)] = 0$ and $\lim_{t\to\infty} \mathrm{Var}(X(t)) = \frac{\sigma^2}{2b}$ and the distribution of the limit random variable $X(\infty)$ is $\mathcal{N}(0, \frac{\sigma^2}{2b})$.
Remark. The limit distribution is still $\mathcal{N}(0, \frac{\sigma^2}{2b})$ if X_0 is a Gaussian random variable independent of the Brownian motion.

The stochastic θ-method applied to the SDE

$$\begin{aligned} \mathrm{d}X(t) &= f(t, X(t))\mathrm{d}t + g(t, X(t))\mathrm{d}W(t), \\ X(0) &= X_0, \end{aligned}$$

is defined for $\theta \in [0,1]$ and a partition $P = \{0 = t_0 < t_1 < ... < t_N = T\}$ of size Δt as

$$X_{n+1} = X_n + f(t_n, X_n)\Delta t(1-\theta) + f(t_{n+1}, X_{n+1})\Delta t\theta + g(t_n, X_n)(W(t_{n+1}) - W(t_n)).$$

For a given $\epsilon > 0$, set $T = 1$, $\sigma = \sqrt{2/\epsilon}$, $b = 1/\epsilon$ and $X_0 = 1$ and apply the θ-method to approximate the solution $X(t)$ of (3.1).

iii) Set $\epsilon = 1/20$. Approximate the solution of equation (3.1) employing the θ-method with $\theta = 0, 1/2, 1$ and uniform partitions $P_k = \{0 = t_0 < t_1 < ... < t_{N_k} = 1\}$ with $N_k = 2^k$ and $k = 2, 4, 6$. Verify that the θ-method with $\theta = 0$ is unstable for large values of Δt.

iv) Consider a uniform partition $P = \{0 = t_0 < t_1 < ... < t_N = 1\}$ with $N = 2^6$. For $\epsilon = 1/20, 1/40, 1/60$ approximate the probability density function f of X_N employing the θ-method with $\theta = 0, 1/2, 1$. Verify that the θ-method with $\theta = 1/2$ is the only one which preserves the limit distribution $N(0, \frac{\sigma^2}{2b})$.
Hint. In order to approximate the density function f of X_N, make a histogram of X_N for $M = 10^4$ independent sample paths and normalize it so that $\int_{\mathbb{R}} f \mathrm{d}x = 1$.

Solution

We have

$$X(t) = X_0 e^{-bt} + \sigma \int_0^t e^{-b(t-s)} \mathrm{d}W(s).$$

Let us prove *ii)* in the case of the remark. Using Itô integral property we have $\mathbb{E}(X(t)) = e^{-bt}\mathbb{E}(X_0)$ and clearly $\lim_{t\to\infty} \mathbb{E}(X(t)) = 0$. Now, using the independence of X_0 and W, we compute

$$\mathbb{E}(X(t)^2) = e^{-2bt}\mathbb{E}(X_0^2) + 2\sigma e^{-bt}\mathbb{E}(X_0)\mathbb{E}\Big(\int_0^t e^{-b(t-s)}\mathrm{d}W(s)\Big) + \sigma^2 \mathbb{E}\Big(\Big(\int_0^t e^{-b(t-s)}\mathrm{d}W(s)\Big)^2\Big).$$

Itô integral property gives $\mathbb{E}(\int_0^t e^{-b(t-s)}\mathrm{d}W(s)) = 0$ and Itô isometry implies

$$\mathbb{E}\Big(\Big(\int_0^t e^{-b(t-s)}\mathrm{d}W(s)\Big)^2\Big) = \int_0^t e^{-2b(t-s)}\mathrm{d}s = \frac{1}{2b}(1 - e^{-2bt}).$$

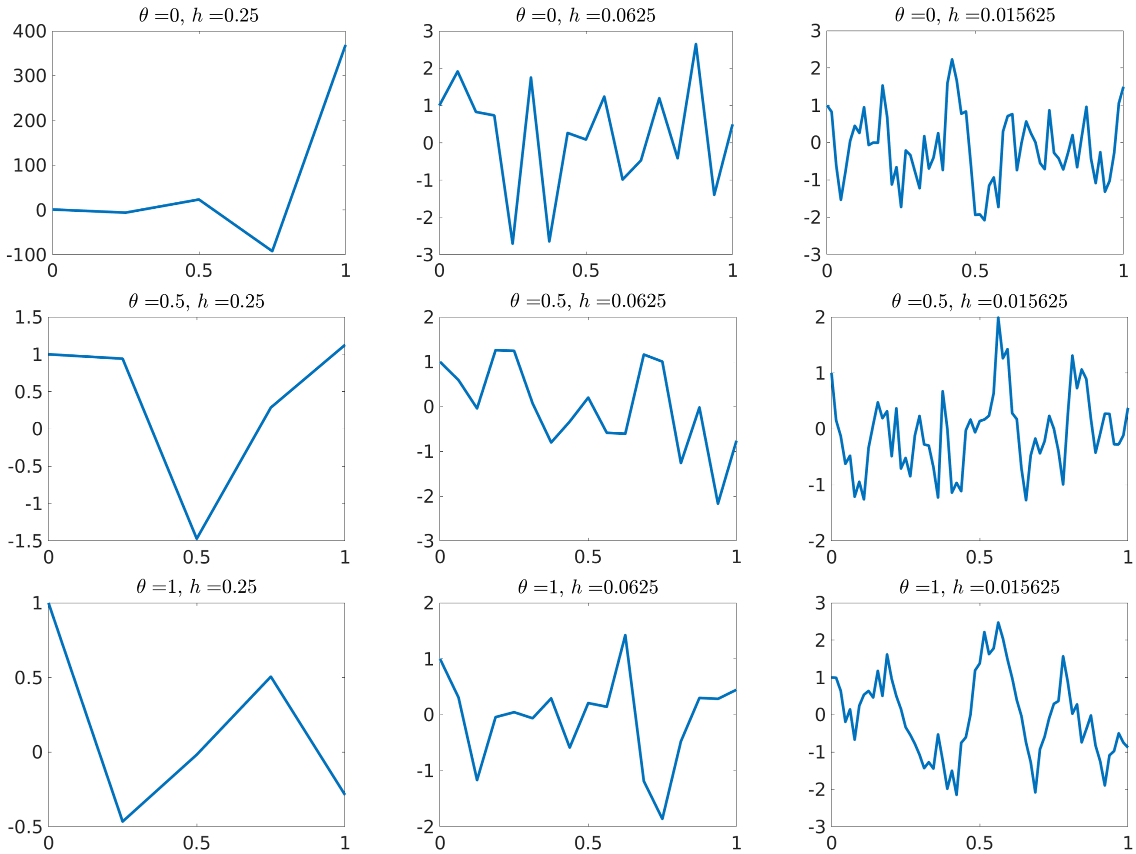

Figure 5: Approximation of the solution of equation (3.1) in Exercise 6 employing the θ-method for different values of θ and Δt.

Hence, we have

$$\mathrm{Var}(X(t)) = \mathbb{E}(X(t)^2) - \mathbb{E}(X(t))^2 = e^{-2bt}\mathbb{E}(X_0^2) + \frac{\sigma^2}{2b}(1 - e^{-2bt}) \overset{t\to\infty}{\longrightarrow} \frac{\sigma^2}{2b}.$$

As an L^2 limit of a Gaussian process, the distribution of the limit random variable X_∞ is $N(0, \frac{\sigma^2}{2b})$. The plots are given in Figures 5 and 6.

Exercise 4.

Consider the SDE

$$\mathrm{d}X(t) = \lambda X(t)\mathrm{d}t + \mu X(t)\mathrm{d}W(t), \tag{4.1}$$

with initial condition $X(0) = X_0$ and where W is a one-dimensional Brownian motion. Show that the fully implicit method

$$X_{n+1} = X_n + \lambda X_{n+1}\Delta t + \mu X_{n+1}\Delta W_n,$$

has unbounded first moments, i.e., $\mathbb{E}[|X_n|] = +\infty$ for all n.

Solution

Notice that

$$X_{n+1} = \frac{1}{1 - \lambda h - \mu \Delta W_n} X_n,$$

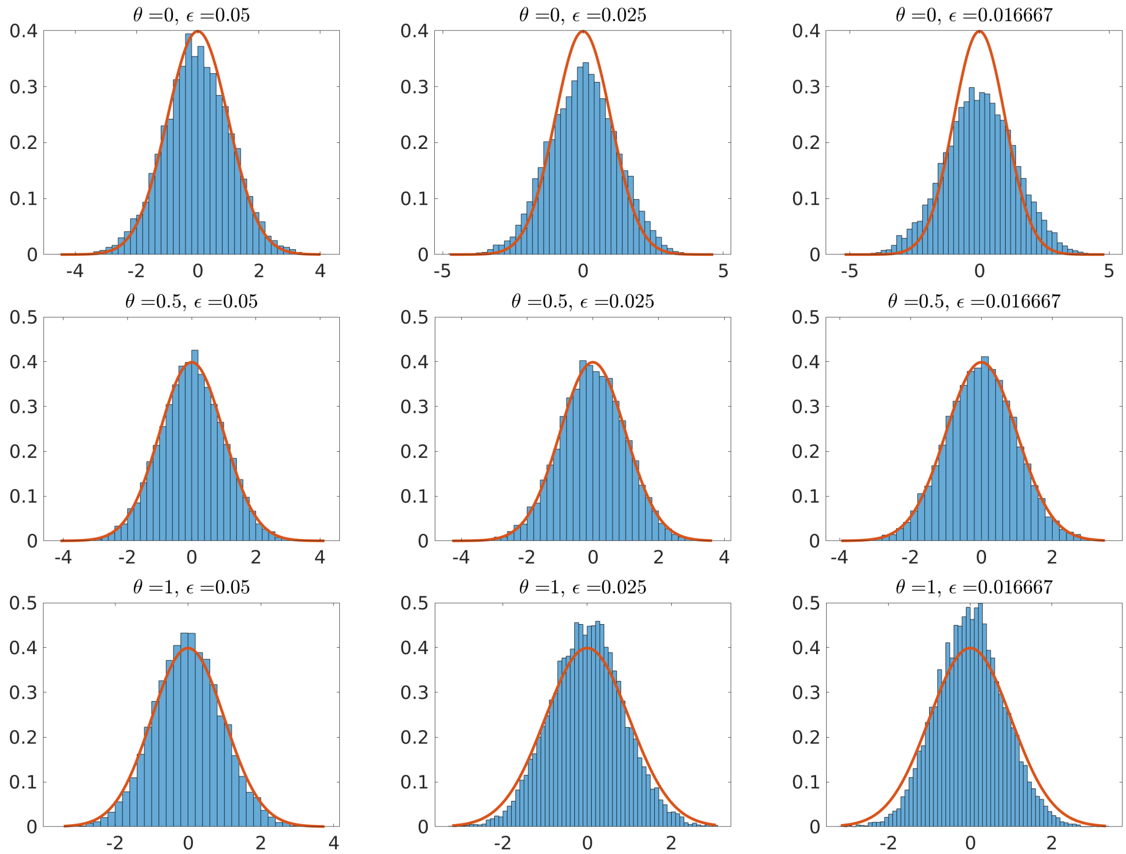

Figure 6: Approximation of the limit distribution employing the θ-method for different values of θ and ϵ.

and define $Z_n = 1/Y_n$ with $Y_n = 1-\lambda h - \mu \varDelta W_n \sim N(m,\sigma^2)$ where $m = 1-\lambda h$ and $\sigma^2 = \mu^2 h$. The sequence $\{Z_n\}_{n=1}^{\infty}$ is independent, hence we have

$$\mathbb{E}|X_{n+1}| = \mathbb{E}|Z_n|\mathbb{E}|X_n| = \mathbb{E}|X_0|\prod_{k=1}^{n}\mathbb{E}|Z_k|.$$

We now show that $\mathbb{E}|Z_n| = \infty$. Set $h(z) = 1/z$ and note that its density function satisfies

$$f_{Z_n}(z) = f_{Y_n}(h(z))|J_h(z)| = f_{Y_n}(1/z)\frac{1}{z^2}.$$

Therefore, for any $\delta > 0$ we have

$$\mathbb{E}|Z_n| = \frac{1}{\sqrt{2\pi}\sigma}\int_{\mathbb{R}}|z|\frac{1}{z^2}e^{-\frac{(1/z-m)^2}{2\sigma^2}}\mathrm{d}z \geqslant \frac{1}{\sqrt{2\pi}\sigma}\int_{\delta}^{\infty}\frac{1}{|z|}e^{-\frac{(1/z-m)^2}{2\sigma^2}}\mathrm{d}z,$$

and choosing δ such that $(1/z-m)^2 \leqslant 2m^2$ for $|z| \geqslant \delta$ we deduce

$$\mathbb{E}|Z_n| \geqslant \frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{m^2}{\sigma^2}}\int_{\delta}^{\infty}\frac{1}{|z|}\mathrm{d}z = \infty.$$

Exercise 5.

Show that the stochastic Heun method

$$\begin{aligned} Y_{n+1} &= Y_n + \frac{1}{2}(b(Y_n,t_n)+b(\bar{Y}_n,t_n))\Delta t + \frac{1}{2}(\sigma(Y_n,t_n)+\sigma(\bar{Y}_n,t_n))\Delta W_n \\ \bar{Y}_n &= Y_n + b(Y_n,t_n)\Delta t + \sigma(Y_n,t_n)\Delta W_n \end{aligned}$$

with fixed time step Δt, i.e. $t_n = n\Delta t$ for all n, is not consistent when applied to the SDE $dX_t = 2X_t dW_t$, $t > 0$, $X_0 = 1$.

Hint: show that $\mathbb{E}[X_t] = 1$, $\forall t$, whereas $\mathbb{E}[Y_n] \not\to 1$ as $\Delta t \to 0$.

Solution

Consider the SDE

$$dX(t) = 2X(t)dW(t),\ t > 0, \qquad X(0) = 1,$$

which we may write in integral form as

$$X(t) = 1 + \int_0^t 2X(s)dW(s).$$

By the martingale property of the Itô integral we obtain

$$\mathbb{E}[X(t)] = 1 + \mathbb{E}\left[\int_0^t 2X(s)dW(s)\right] = 1 + 0 = 1 \quad \text{for all } t \geqslant 0. \tag{17.5}$$

The stochastic Heun method here is

$$Y_{n+1} = Y_n + \frac{1}{2}\Delta W_n[2Y_n + 2(Y_n + 2\Delta W_n Y_n)] = Y_n[1 + 2\Delta W_n + 2(\Delta W_n)^2].$$

Hence

$$Y_n = \prod_{j=0}^{n-1}\big(1 + 2\Delta W_j + 2(\Delta W_j)^2\big).$$

Since the factors are independent, on taking expectations we have

$$\mathbb{E}[Y_n] = \prod_{j=0}^{n-1}\mathbb{E}\big[1 + 2\Delta W_j + 2(\Delta W_j)^2\big] = \prod_{j=0}^{n-1}(1 + 2\Delta t) = (1 + 2\Delta t)^n. \tag{17.6}$$

If we consider the limit $\Delta t \to 0$ for fixed $t_n = n\Delta t$, then

$$\mathbb{E}[Y_n] = e^{2t_n} + O(\Delta t). \tag{17.7}$$

So the stochastic Heun method does not converge weakly (and also not strongly) for this simple example. Therefore, the method is not consistent.