MATH-450
Numerical integration of stochastic differential equations

Prof. Fabio Nobile

A.Y. 2024-2025 – Fall semester

Course organization

- Lectures ex cathedra (blackboard):
  - **Monday** 13:15 – 15:00, room **CO 121**
- Exercice sessions (theoretical and computer based exercises):
  - **Wednesday** 13:15 – 15:00, room **DIA 004**
  - The assigned room is not equipped with computers. For the computer based exercises **bring your own laptop**.
  - **No exercise session this week**
- Assistant
  - Fabio Zoccolan (`fabio.zoccolan@epfl.ch`)
  - Office hours Tuesdays 14:15-15:00 (tentative), office MA B2 435. Contact by email before going.
- Relevant material for the course available on the *moodle* web site

Why numerical methods for SDEs?

$$dX_t = b(X_t, t)dt + \sigma(X_t, t)dW_t$$

- SDE models appear in many applications: Finance, statistical physics, Biology (population dynamics, epidemiology, etc.), Chemical reaction systems, Machine Learning, etc.
- Exact solutions are available only for simple models. In more complext situations, numerical methods are needed.
- It is important to analyze and control the accuracy and stability of numerical methods: standard ones for Ordinary Differential Equations (ODEs) taught in Numerical Analysis courses may not work and have to be properly adjusted to the SDE case.

Examples of SDEs in applications

Asset price model in finance

A widely used model to describe the evolution of the price S_t of a certain asset is given by the so called *Geometric Brownian Motion*

$$dS_t = rS_t dt + \sigma S_t dW_t$$

where W_t is a Brownian Motion.

Ofter the goal is to compute quantities like $e^{rT}\mathbb{E}[\phi(S_T)\mathbb{I}_{\tau>T}]$ (option pricing) possibly involving some stopping time $\tau = \inf\{t : S_t \leq b\}$

The model can be complicated in many directions, e.g.

- replacing W_t with other stochastic processes, Jump processes, Levy processes, fractional Brownian Motion, ...
- considering stochastic volatility models, where $v = \sigma^2$ satisfies another SDE, e.g. a CIR process

$$dv_t = k(\theta - v_t)dt + \xi\sqrt{v_t}\widetilde{W}_t$$

with $\widetilde{W}_t$ another Brownian motion correlated with W_t.

Examples of SDEs in applications

Wright-Fischer diffusion model for population genetics

The Wright-Fischer model is a discrete Markov Chain model describing the frequence of alleles of a gene in a population. The simplest version is for an haploid population with only two alleles A_1 and A_2.

Each individual of a new generation inherits the allele randomly from an individual of the old generation. The model with mutation allows for a random switch of allele with a certain probability right after birth.

For a large population and lond time horizon, the Markov chain can be approximated by an SDE

$$dX_t = [-\alpha X_t + \beta(1 - X_t)]dt + \sqrt{\gamma X_t(1 - X_t)}dW_t$$

where $X_t \in [0, 1]$ is the relative frequence of Allele A_1 (or A_2) and α β are the mutation rates $A_1 \to A_2$ and $A_2 \to A_1$, respectively.

Examples of SDEs in applications

Chemical reaction network

Consider a system of N chemical species $S_1, \ldots, S_N$ and M chemical reactions, e.g. the j-th reaction may look like

$$S_1 + S_3 \xrightarrow{c_j} 2S_2 + S_4$$

It can be encoded in the stoichiometric vector $\boldsymbol{\nu}_j = (-1, 2, -1, 1, 0, \ldots)$ and propensity function (reaction rate) $\alpha_j(\mathbf{X}) = c_j X_1 X_3$ where $\mathbf{X} = (X_1, \ldots, X_N)$ is the vector of concentrations (or number of molecules) of each species.

The evolution of $\mathbf{X}_t$ over time can be described by a time continuous discrete space Markov chain (Jump process – the state changes discontinuously every time a reaction happens)

However, for large number of molecules (hence very frequent reactions) the jump process can be approximated by a diffusion leading a system of SDEs (chemical Langevin equation)

$$d\mathbf{X}_t = \sum_{j=1}^{M} \boldsymbol{\nu}_j a_j(\mathbf{X}_t)dt + \sum_{j=1}^{M} \boldsymbol{\nu}_j \sqrt{a_j(\mathbf{X}_t)}dW_y^j$$

EPFL

SDEs and PDEs

There is a beautiful link between SDEs and PDEs via the Feynman-Kac formula

E.g., consider the one-dimensional SDE

$$dX_t = b(X_t, t)dt + \sigma(X_t, t)dW_t$$

and the (backward in time) PDE

$$\begin{cases} \partial_t u(x,t) + b(x,t)\partial_x u(x,t) + \frac{1}{2}\sigma^2(x,t)\partial_{xx} u(x,t) = 0, & (x,t) \in \mathbb{R} \times [0,T) \\ u(x,T) = \phi(x) \end{cases}$$

The Feynman-Kac formula says that $u(x,s) = \mathbb{E}^{x.s}[\phi(X_T)]$ where $\mathbb{E}^{x,t}$ means taking expectation when the process X_t is started in x at time s.

This link can be used to compute the solution of the PDE in one point (or few points) by solving the corresponding SDE multiple times and averaging the result. Appealing is high dimension.

Contents of the course

- Review of stochastic calculus
  - Brownian motion, Stochastic integral; Ito's formula, stochastic differential equations; Feynman-Kac's formula
- Numerical methods for stochastic differential equations
  - Euler Maruyama scheme; strong and weak convergence; stability; Milstein scheme and other integrators, Multi-level Monte-Carlo methods
- Other topics that may be addressed if time permits
  - numerical integration of non-Lipschits SDEs; approximation of mean exit time and stopped diffusion; long time integration and approximation of invariant measure; numerical integration of McKean Valsov SDEs, jump diffusion processes, Backward SDEs.

Prerequisits: Numerical Analysis, Advanced probability,

Recommended background: Stochastic calculus

Reference books

- **References for numerical methods**
  - P.E. Kloeden, E. Platen, "Numerical Solution of Stochastic Differential Equations", Springer, 1999.
  - G.N. Milstein, M.V. Tretyakov, "Stochastic Numerics for Mathematical Physics", Springer, 2004.
  - D. Higham, P. Kloeden, "An Introduction to the Numerical Simulation of Stochastic Differential Equations", SIAM 2021
- **References for stochastic calculus**
  - L.C. Evans, "An Introduction to Stochastic Differential Equations", AMS, 2013.
  - B. Øksendal, " Stochastic Differential Equations. An Introduction with Applications", Springer, 2003
  - H-H. Kuo, "Introduction to Stochastic Integration", Springer, 2005.
  - L. Arnold, "Stochastic Differential Equations, Theory and applications", Dover Publications, 1974.

I'm (hand)writing some notes. If they will be decently readable, they will be shared on Moodle.

Exercise sessions

- The text of the exercises will be available on the moodle web page one day before the exercise session.
- Solutions of some (but not all) of the exercises will be made available on moodle one week after the exercise session.
- The exercises will be a mix of theory and practice on computer. (Bring your own laptop)
- Solutions to the computer based exercises will be provided in Matlab. However, you are free to use Python if you prefer.

Exam

- The final grade is based on a project (30% of the grade) and a written exam (70% of the grade)
- A list of projects will be given by the end of September. A report has to be handed in at the end of the semester (Before Christmas)
- The project can be done individually or in group of 2 at most
- The written exam will be in a computer room and will include theoretical questions as well as questions which require the use of the computer.
- Both the project and the computer based questions at the exam can be solved in Matlab or Python.

Questions?