

Series 9 - November 20, 2024

Exercise 1.

- 1) Show that the following nonlinear SDE

$$dX_t = dt + 2\sqrt{X_t}dW_t \quad (1.1)$$

has solution $X_t = (W_t + \sqrt{X_0})^2$ for $t \leq \tau = \inf\{t > 0 : W_t < -\sqrt{X_0}\}$.

- 2) Show that the following nonlinear SDE

$$dX_t = \left(\sqrt{1 + X_t^2} + \frac{1}{2}X_t \right) dt + \sqrt{1 + X_t^2} dW_t \quad (1.2)$$

has solution $X(t) = \sinh(\operatorname{arcsinh}(X_0) + t + W_t)$.

Exercise 2.

Consider the modified Euler–Maruyama method given by

$$X_{n+1} = X_n + f(t_n, X_n)h + g(t_n, X_n)\xi_n,$$

where $\{\xi_n\}_{n \geq 0}$ is a sequence of independent random variables such that ξ_n is independent of X_n for all n and

$$\mathbb{E}[\xi_n] = 0, \quad \mathbb{E}[\xi_n^2] = h, \quad \mathbb{E}[\xi_n^3] = 0, \quad |\mathbb{E}[\xi_n^4]| = o(h^2).$$

Since $\mathbb{E}[\xi_n^\ell] = \mathbb{E}[\Delta W_n^\ell]$ for $\ell = 1, 2, 3$ and $\mathbb{E}[\xi_n^4]$ is bounded, then it is possible to prove that this method has weak order 1.

- i) Give an example of discrete random variables ξ_n satisfying the hypotheses above.
- ii) Verify numerically that this method has weak order 1. Set $f(x) = \lambda x$ with $\lambda = 2$, $g(x) = \mu x$ with $\mu = 0.1$, $X_0 = 1$ and $T = 1$. Choose different step sizes $h = 2^{-i}$ with $i = 4, \dots, 10$ and approximate the expectations of the weak error via using $M = 10^4$ realizations of $\{\xi_n\}_{n \geq 0}$.

Exercise 3.

Consider the following geometric Brownian motion

$$\begin{aligned} dX_t &= \mu X_t dt + \sigma X_t dW_t, \quad t \in [0, T], \\ X(0) &= X_0 \in \mathbb{R} \end{aligned} \quad (3.1)$$

with $\mu, \sigma \in \mathbb{R}$. Consider a uniform partition $P = \{0 = t_0 < t_1 < \dots < t_N = T\}$ of size Δt . For the linear SDE (3.1), the Euler-Maruyama method produces the recurrence

$$Y_{n+1} = (1 + \mu \Delta t + \sigma \Delta W_n) Y_n, \quad n = 0, \dots, N-1. \quad (3.2)$$

1) Noting that ΔW_n is independent of Y_n , take expected values to show that

$$\mathbb{E}[Y_{n+1}] = (1 + \mu\Delta t)\mathbb{E}[Y_n].$$

Considering the limit $\Delta t \rightarrow 0$ and $N \rightarrow \infty$ with $N\Delta t = T$ fixed, show that

$$\mathbb{E}[Y_N] \rightarrow e^{\mu T}\mathbb{E}[X_0], \quad N \rightarrow \infty$$

2) Show that

$$\mathbb{E}[Y_{n+1}^2] = ((1 + \mu\Delta t)^2 + \sigma\Delta t)\mathbb{E}[Y_n^2]$$

and, hence under $\Delta t \rightarrow 0$ and $N \rightarrow \infty$ with $N\Delta t = T$, one has that

$$\mathbb{E}[Y_N^2] \rightarrow e^{(2\mu + \sigma^2)T}\mathbb{E}[X_0^2], \quad N \rightarrow \infty.$$

3) For $\mu = 2$, $\sigma = 0.1$, $X_0 = 1$, $T = 1$, $M = 10^4$, plot the weak error $|\mathbb{E}[Y_N] - \mathbb{E}[X_T]|$.

Exercise 4.

Consider the following Langevin dynamics

$$\begin{aligned} dX_t &= \mu X_t dt + \sigma dW_t, \quad t \in [0, T], \\ X(0) &= X_0 \in \mathbb{R} \end{aligned} \tag{4.1}$$

with $\mu, \sigma \in \mathbb{R}$. Consider a uniform partition $P = \{0 = t_0 < t_1 < \dots < t_N = T\}$ of size Δt . For the linear SDE (4.1), the Euler-Maruyama method produces the following recurrence

$$Y_{n+1} = (1 - \mu\Delta t)Y_n + \sigma\Delta W_n. \tag{4.2}$$

Repeat the same computations of Exercise 3.

Exercise 5.

Let us recall that if we consider an Itô stochastic differential equation (here one dimensional for simplicity)

$$dX_t = a(t, X_t)dt + b(t, X_t)dW_t$$

whose solution is X_t , then, X_t solves the following Stratonovich SDE

$$dX_t = \underline{a}(t, X_t)dt + b(t, X_t) \circ dW_t,$$

where $\underline{a}(t, x) = a(t, x) - \frac{1}{2}b(t, x)\frac{\partial b}{\partial x}(t, x)$. On the contrary, given a Stratonovich SDE

$$dX_t = \underline{a}(t, X_t)dt + b(t, X_t) \circ dW_t,$$

then the corresponding Itô differential equation is

$$dX_t = \left(\underline{a}(t, x) + \frac{1}{2}b(t, x)\frac{\partial b}{\partial x}(t, x) \right) dt + b(t, X_t)dW_t.$$

Let $\lambda, \mu \in \mathbb{R}$ and consider the SDE for $t \in [0, T]$

$$\begin{aligned} dX(t) &= \lambda X(t)dt + \mu X(t)dW(t), \\ X(0) &= X_0. \end{aligned} \tag{5.1}$$

The solution of (5.1) in the Itô sense is given by $X(t) = X_0 e^{(\lambda - \frac{1}{2}\mu^2)t + \mu W(t)}$.

Let $P = \{0 = t_0 < t_1 < \dots < t_m = T\}$ be a partition of $[0, T]$ of size Δt and define the Euler polygonal interpolant $\widehat{W}$ of W on P as

$$\widehat{W}(t) = W(t_{n-1}) + (W(t_n) - W(t_{n-1})) \frac{t - t_{n-1}}{t_n - t_{n-1}}, \quad 1 \leq n \leq m, \quad t_{n-1} \leq t \leq t_n.$$

If we replace W by $\widehat{W}$ in the SDE (5.1), we obtain the ordinary differential equation

$$\begin{aligned} \frac{d}{dt} \widehat{X}^m(t) &= \lambda \widehat{X}^m(t) + \mu \widehat{X}^m(t) \frac{d}{dt} \widehat{W}(t), \\ \widehat{X}^m(0) &= X_0. \end{aligned} \tag{5.2}$$

iv) Compute the solution $\widehat{X}^m(t)$ of (5.2).

v) What is the limit in $L^2(\Omega)$ of $\widehat{X}^m(t)$ as $m \rightarrow \infty$ (i.e., as $\Delta t \rightarrow 0$)?

In order to approximate numerically the solution $\widehat{X}^m$ of (5.2) we can use the following scheme

$$\widehat{X}^m(t_{n+1}) = \widehat{X}^m(t_n) + \lambda \widehat{X}^m(t_n) \Delta t + \mu (W(t_{n+1}) - W(t_n)) \frac{1}{\Delta t} \int_{t_n}^{t_{n+1}} \widehat{X}^m(s) ds, \tag{5.3}$$

where we need to approximate the integral $\int_{t_n}^{t_{n+1}} \widehat{X}^m(s) ds$.

vi) If we approximate the integral with the Euler formula

$$\int_{t_n}^{t_{n+1}} \widehat{X}^m(s) ds \approx \Delta t \widehat{X}^m(t_n),$$

what method do we obtain?

vii) We now approximate the integral with the trapezoidal rule

$$\int_{t_n}^{t_{n+1}} \widehat{X}^m(s) ds \approx \frac{\Delta t}{2} (\widehat{X}^m(t_n) + \widehat{X}^m(t_{n+1})),$$

and make an Euler prediction for the implicit term

$$\widehat{X}^m(t_{n+1}) \approx \widehat{X}^m(t_n) + \lambda \widehat{X}^m(t_n) \Delta t + \mu \widehat{X}^m(t_n) (W(t_{n+1}) - W(t_n)).$$

Write the method derived from these approximations.

viii) Let $T = 1$, $\lambda = 2$, $\mu = 1$, $X_0 = 1$ and consider a uniform partition of $[0, T]$ with $\Delta t = 10^{-2}$. Implement the numerical methods derived in points *vi)* and *vii)*. What solutions do these methods converge to? What are the strong orders of convergence of the methods? In order to observe numerically the strong order, plot the error for different values of $\Delta t = 2^{-i}$ with $i = 4, 5, \dots, 11$ employing $M = 10^4$ different Brownian paths.