Series 8 - November 13, 2024

Exercise 1.

Let $\{W_t\}_t$ be an m-dimensional Brownian motion. We recall the definitions

$$I_n(k,l) = \int_{t_n}^{t_{n+1}} \int_{t_n}^{s} dW_u^k dW_s^l, \quad I_n(k,l,j) = \int_{t_n}^{t_{n+1}} \int_{t_n}^{s} \int_{t_n}^{u} dW_r^k dW_u^l dW_s^j,$$

with the convention $W_t^0 = t$ (hence $dW_t^0 = dt$). Show that

1) $I_n(0,0) = \frac{1}{2}\Delta t^2$,
2) $I_n(0,k) = \Delta t \Delta W_n^k - I_n(k,0)$
3) $I_n(k,0) = \frac{1}{2}\Delta t(\Delta W_n^k + \frac{1}{\sqrt{3}} Z_k)$ with $Z^k \sim N(0,\Delta t)$ independent of ΔW_n^k.
4) $I_n(k,k,k) = \frac{1}{2}(\frac{1}{3}(\Delta W_n^k)^2 - \Delta t)\Delta W_n^k$

Exercise 2.

Let $W_1, \dots, W_m$ be m independent one-dimensional Brownian motions in $[0,T]$. Consider a uniform partition $P = \{0 = t_0 < t_1 < \dots < t_N = T\}$ of size Δt. We recall that

$$I_n(i_p, \dots, i_1) := \int_{t_n}^{t_{n+1}} \int_{t_n}^{s_1} \dots \int_{t_n}^{s_{p-1}} \mathrm{d}W_{s_p}^{i_p} \dots \mathrm{d}W_{s_1}^{i_1}$$

where for $\mathrm{d}W_s^{i_j} = 0$if $i_j = 0$ For example, we have

$$I_n(1,2,0) = \int_{t_n}^{t_{n+1}} \int_{t_n}^{s_1} \int_{t_n}^{s_2} \mathrm{d}W_{s_3}^1 \mathrm{d}W_{s_2}^2 \mathrm{d}s_1.$$

Show that:

i) $\mathbb{E}[I_n(i_p, \dots, i_1)] = 0$ if there exists $\hat{j} \in \{1, \dots, p\}$ such that $i_{\hat{j}} \neq 0$.
ii) $\mathbb{E}[I_n(i_p, \dots, i_1)] = \mathcal{O}((\Delta t)^p)$ if $i_j = 0$ for all $j \in \{1, \dots, p\}$.
Hint. Use the Cauchy formula for repeated integration.
iii) $\mathbb{E}[I_n^2(i_p, \dots, i_1)]^{1/2} = \mathcal{O}((\Delta t)^q)$ where $q = \sum_{j=1}^p \frac{2-\min\{1,i_j\}}{2}$.
Hint. Prove it by induction.
iv) $\mathbb{E}[|I_n(i_p, \dots, i_1)|] = \mathcal{O}((\Delta t)^q)$ where $q = \sum_{j=1}^p \frac{2-\min\{1,i_j\}}{2}$.

Exercise 3.

The aim of this exercise is to derive a method of strong order 3/2 for an autonomous one-dimensional SDE of the form

$$\mathrm{d}X(t) = f(X(t))\mathrm{d}t + g(X(t))\mathrm{d}W(t).$$

Let us first define the following Itô integrals with the notation of Exercise 3

$$I_n(0) = \int_{t_n}^{t_{n+1}} \mathrm{d}s_1, \qquad I_n(1) = \int_{t_n}^{t_{n+1}} \mathrm{d}W(s_1),$$

$$I_n(0,0) = \int_{t_n}^{t_{n+1}} \int_{t_n}^{s_1} \mathrm{d}s_2 \mathrm{d}s_1, \qquad I_n(0,1) = \int_{t_n}^{t_{n+1}} \int_{t_n}^{s_1} \mathrm{d}W_{s_2} \mathrm{d}s_1,$$

$$I_n(1,0) = \int_{t_n}^{t_{n+1}} \int_{t_n}^{s_1} \mathrm{d}s_2 \mathrm{d}W_{s_1}, \qquad I_n(1,1) = \int_{t_n}^{t_{n+1}} \int_{t_n}^{s_1} \mathrm{d}W_{s_2} \mathrm{d}W_{s_1},$$

$$I_n(1,1,1) = \int_{t_n}^{t_{n+1}} \int_{t_n}^{s_1} \int_{t_n}^{s_2} \mathrm{d}W_{s_3} \mathrm{d}W_{s_2} \mathrm{d}W_{s_1}, \quad \dots$$

In the following, we denote by f, f', f'' and g, g', g'' the values $f(X(t_n)), f'(X(t_n)), f''(X(t_n))$ and $g(X(t_n)), g'(X(t_n)), g''(X(t_n)$ respectively. Moreover, we define

$$e_1 = \int_{t_n}^{t_{n+1}} f(X(s_1))\mathrm{d}s_1 - f\Delta t,$$
$$e_2 = \int_{t_n}^{t_{n+1}} g(X(s_1))\mathrm{d}W_{s_1} - g\Delta W_n.$$

i) Show that

$$e_1 = f'gI_n(0,1) + (ff' + \frac{1}{2}g^2f'')I_n(0,0) \\ + \mathcal{O}(I_n(0,1,0)) + \mathcal{O}(I_n(0,1,1)) + \mathcal{O}(I_n(0,0,0)) + \mathcal{O}(I_n(0,0,1)).$$

ii) Show that

$$e_2 = (fg' + \frac{1}{2}g^2g'')I_n(1,0) + gg'I_n(1,1) + g(gg'' + (g')^2)I_n(1,1,1) \\ + \mathcal{O}(I_n(1,0,0)) + \mathcal{O}(I_n(1,0,1)) + \mathcal{O}(I_n(1,1,0)) + \mathcal{O}(I_n(1,1,1,0)) + \mathcal{O}(I_n(1,1,1,1)).$$

Then, a possible proposal for a method of strong order 3/2 is given by

$$Y_{n+1} = Y_n + f\Delta t + g\Delta W_n + f'gI_n(0,1) + \left(ff' + \frac{1}{2}g^2f''\right)I_n(0,0) \\ + \left(fg' + \frac{1}{2}g^2g''\right)I_n(1,0) + gg'I_n(1,1) + g(gg'' + (g')^2)I_n(1,1,1). \tag{3.1}$$

iii) Using the results of Exercise 1, write explicitly the method (3.1).

Exercise 4.

Let $a \in \mathbb{R}$ and consider the SDE on $[0,T]$

$$\mathrm{d}X(t) = -\frac{1}{2}a^2X(t) + a\sqrt{1 - X(t)^2}\mathrm{d}W(t), \tag{4.1}$$

with initial condition $X(0) = X_0$.

i) Verify that the Itô solution of equation (4.1) is given by

$$X(t) = \sin(aW(t) + \arcsin(X_0)).$$

ii) Write the Milstein scheme for equation (4.1) and the corresponding derivative-free schemes with supporting values

$$\begin{aligned} \widehat{Z}_n &= X_n + f(t_n, X_n)\Delta t + g(t_n, X_n)\sqrt{h}, \\ \widetilde{Z}_n &= X_n + g(t_n, X_n)\sqrt{\Delta t}. \end{aligned} \tag{4.2}$$

iii) Implement the schemes above and verify that they all have strong order of convergence 1. Set $a = 0.5$, $X_0 = 0$, $T = 1$. Choose different step sizes $\Delta t = 2^{-i}$ with $i = 2, \dots, 8$ and approximate the expectations using $M = 10^4$ realizations of the Brownian motion. What difference do you notice between the three schemes?

iv) Repeat the previous point for the strong order 1.5 scheme of Exercise 4.

Exercise 5.

Consider the SDE on $[0, 1]$

$$\mathrm{d}X(t) = -\frac{1}{2}X(t)\mathrm{d}t + X(t)\mathrm{d}W_1(t) + X(t)\mathrm{d}W_2(t),$$

with initial condition $X(0) = 1$. Does this problem have commutative noise? Give the exact solution and compute the numerical solution employing the Milstein–Platen scheme. Verify that it has strong order of convergence 1. Choose different step sizes $\Delta t = 2^{-i}$ with $i = 2, \dots, 8$ and approximate the expectations using $M = 10^4$ realizations of the Brownian motion.