Series 6 - October 30, 2024

Exercise 1.

(The maximum principle) Consider a d-dimensional SDE driven by an m-dimensional Brownian motion with $m \geqslant d$.

$$dX_t = b(X_t)dt + \sigma(X_t)dB_t, \quad t \geqslant 0$$

and its corresponding generator

$$L = \frac{1}{2}\sum_{i,j=1}^{d} a_{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{d} b_i(x)\frac{\partial}{\partial x_i}$$

with $a_{ij} = \sum_{l=1}^{m} \sigma_{il}\sigma_{jl}$. Let $D \subset \mathbb{R}^d$ be a bounded domain and suppose b, σ globally Lipschitz and satisfying the linear growth bound. Moreover assume a uniformly elliptic on $\bar{D}$ (i.e. $\exists \lambda > 0$ such that $v^T a(x) v \geqslant \lambda |v|^2$ for all $x \in \bar{D}$, $v \in \mathbb{R}^d$). Let $u \in C^2(D) \cap C(\bar{D})$. Using the Feynman-Kac formula, show the maximum principle, i.e.

- if $Lu \geqslant 0$, then $u(x) \leqslant \max\limits_{\partial D} u$ for all $x \in D$
- if $Lu = 0$, then $\min\limits_{\partial D} u \leqslant u(x) \leqslant \max\limits_{\partial D} u$ for all $x \in D$

Solution

Let $\tau = \inf\{t \ : \ X_t^{x,0} \notin D\}$, where $X_t^{x,0}$ is the process that starts in x at time zero, and set $f = Lu$. Then

$$u(x) = \mathrm{E}\Big[\varphi\Big(X_\tau^{x,0}\Big)\Big] - \mathrm{E}\bigg[\int_0^\tau f\Big(X_s^{x,0}\Big)ds\bigg].$$

Hence, if $f \geqslant 0$, then we have

$$u(x) \leqslant \mathrm{E}\Big[\varphi\Big(X_\tau^{x,0}\Big)\Big] \leqslant \max_{\partial D} u,$$

If $f = 0$, then $f \geqslant 0$ and $f \leqslant 0$, hence

$$\min_{\partial D} u \leqslant u(x) \leqslant \max_{\partial D} u.$$

Exercise 2.

Consider a d-dimensional SDE driven by an m-dimensional Brownian motion with $m \geqslant d$.

$$dX_t = b(X_t)dt + \sigma(X_t)dB_t, \quad t \geqslant 0 \tag{2.1}$$

and its corresponding generator

$$L = \frac{1}{2}\sum_{i,j=1}^{d} a_{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{d} b_i(x)\frac{\partial}{\partial x_i}$$

with $a_{ij} = \sum_{l=1}^{m} \sigma_{il}\sigma_{jl}$. Let $D \subset \mathbb{R}^d$ be a bounded open domain and suppose b, σ globally Lipschitz and satisfying the linear growth bound. Moreover assume a uniformly elliptic on $\bar{D}$ (i.e. $\exists \lambda > 0$ such that $v^T a(x) v \geqslant \lambda |v|^2$ for all $x \in \bar{D}$, $v \in \mathbb{R}^d$).

1) Show that if $u \in C^2(D) \cap C(\bar{D})$ is the solution of

$$\begin{cases} Lu = -1 & \text{on } D \\ u_{|_{\partial D}} = 0, \end{cases} \tag{2.2}$$

then u has the characterization $u(x) = \mathbb{E}^x(\tau)$ where $\tau = \inf\{t \;:\; X_t \notin D\}$ and $\mathbb{E}^x$ denotes expectation when the process (2.1) starts in x at time zero.

2) Let $\{B_t\}_t$ be a one dimensional Brownian motion and $\tau = \inf\{t \;:\; B_t \notin (-1, 1)\}$. Compute $\mathbb{E}[\tau]$.

3) Let

$$\begin{aligned} dX_t &= \sigma(X_t) dB_t \\ X_0 &= x \end{aligned} \tag{2.3}$$

with σ globally Lipschitz on $\mathbb{R}$ and strictly positive in $[0, 1]$. Let $\tau = \inf\{t \;:\; X_t \notin (0, 1)\}$ be the exit time from $(0, 1)$. Show that $\tau < \infty$ a.s. and compute $P(X_\tau = 1)$ (notice that this probability does not depend on σ).

Solution

1) Just apply the Feynman-Kac formula, recalling that here $\phi = 0, c = 0$ and $f = -1$.

2) Let $d = 1$ and $\mathrm{d}X_t = \mathrm{d}B_t$ then (2.2) becomes

$$\begin{cases} \frac{1}{2} u'' = -1 \\ u(-1) = u(1) = 0 \end{cases}$$

and has the solution $u(x) = 1 - x^2$. Hence $\mathbb{E}[\tau] = u(0) = 1$.

3) Under the assumptions on σ, (2.2) in $(0, 1)$ has a unique solution $u \in C^2((0, 1)) \cap C([0, 1])$ and is non negative from maximum principle (see Exercise 1). From point 1) we have that $\mathbb{E}^x[\tau] = u(x) < \infty$ for all $x \in D$, hence τ is a.s. bounded. To compute $P(X_\tau = 1)$ one has two possibilities. The former consists to see that $u(x) = P(X_\tau = 1)$ is solution of

$$\begin{cases} \frac{1}{2} \sigma^2(x) u''(x) = 0 & 0 < x < 1 \\ u(0) = 0, \;\; u(1) = 1, \end{cases} \tag{2.4}$$

As $\sigma^2(x)$ is always strictly positive, the problem is equivalent to $u'' = 0$ with the above boundary condition and, hence, its solution does not depend on the diffusion. Therefore, the solution is $u(x) = P(X_\tau = 1) = x$.

Otherwise, one could have notice that

$$X_{t\wedge\tau} = x + \int_0^{t\wedge\tau} \sigma(X_s) dB_s$$

is a martingale. Therefore $\mathbb{E}^x[X_{t\wedge\tau}] = x$. As $|X_{t\wedge\tau}| \leqslant 1$, then we can use Lebesgue's theorem and passing to the limit as $t \to +\infty$ one gets $x = \mathbb{E}^x[X_\tau]$. As X_τ can have values 0 or 1, we get $P(X_\tau = 1) = x$.

Exercise 3.

Let $\lambda = 2$, $\mu = 1$ and consider the stochastic differential equation

$$\begin{aligned} &dX(t) = \lambda X(t)dt + \mu X(t)dW(t), \quad 0 \leqslant t \leqslant T, \\ &X(0) = X_0, \end{aligned} \tag{3.1}$$

and the Euler–Maruyama (EM) method for (3.1)

$$X_{n+1} = X_n + \lambda X_n \Delta t + \mu X_n (W(t_{n+1}) - W(t_n)).$$

The exact solution of (3.1) is given by $X(t) = X_0 \exp((\lambda - \frac{1}{2}\mu^2)t + \mu W(t))$.
Compute a discretized Brownian path over $[0, 1]$ with $\delta t = 2^{-8}$ and compare the exact solution (on the discretized path) with the EM method (using the same Brownian path) with $\Delta t = 2^4 \delta t, 2^2 \delta t$.

Solution

The plot is given in Figure 1.

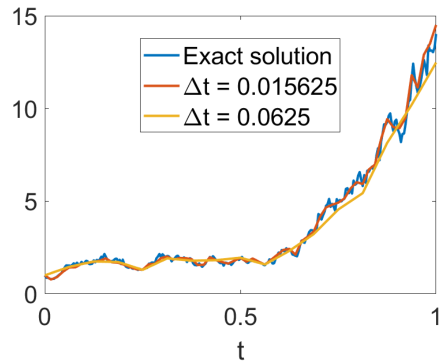

Figure 1: EM approximation of the SDE in Exercise .

Exercise 4.

A numerical method for an SDE

$$\begin{aligned} &dX(t) = f(X(t))dt + g(X(t))dW(t), \quad 0 \leqslant t \leqslant T, \\ &X(0) = X_0, \end{aligned} \tag{4.1}$$

is said to have a *strong order of convergence equals to r* if there exists a constant C such that

$$\mathbb{E}[\sup_{0 \leqslant n \leqslant N} |X_n - X(t_n)|] \leqslant C(\Delta t)^r,$$

with $N = T/\Delta t$. Consider the following one dimensional SDE

$$\begin{aligned} dX(t) &= \lambda X(t)dt + \mu X(t)dW(t), \quad 0 \leqslant t \leqslant T, \\ X(0) &= X_0, \end{aligned} \tag{4.2}$$

and set $t_n = T = 1$. Consider $\{X_n\}_{n=0}^N$ the Euler-Maruyama approximation and set $e^{\mathrm{s}}_{\Delta t} := \mathbb{E}|X_N - X(T)|$. Verify numerically that the Euler-Maruyama method satisfies $e^{\mathrm{s}}_{\Delta t} \leqslant C(\Delta t)^{1/2}$. To evaluate $\mathbb{E}|X_N - X(T)|$ you need to compute $\frac{1}{M}\sum_{i=1}^M |X^i_N - X^i(T)|$, i.e., the average over M realizations of the random variables at time $T = 1$. For that:

i) take $M = 10^5$ independent discretized Brownian path over $[0, 1]$ with $\delta t = 2^{-10}$,
ii) for each path apply EM with $\Delta t = 2^p \delta t$, $1 \leqslant p \leqslant 5$ and store the endpoint error (at $t = T$),
iii) take the mean over the error and then report the result (Δt versus $e^{\mathrm{s}}_{\Delta t}$) in a loglog plot.

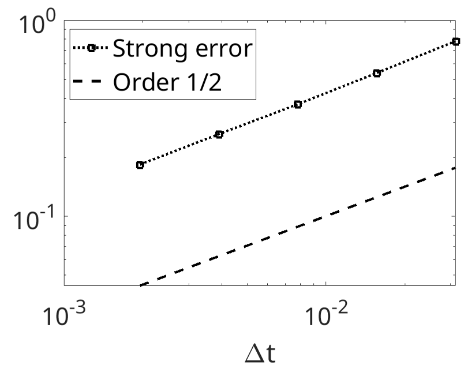

Figure 2: Strong order of convergence 1/2 for the EM method.

Solution

The plot is given in Figure 2.

Exercise 5.

A numerical method for an SDE

$$\begin{aligned} &dX(t) = f(X(t))dt + g(X(t))dW(t), \quad 0 \leqslant t \leqslant T, \\ &X(0) = X_0, \end{aligned} \tag{5.1}$$

is said to have a *weak order of convergence equals to* r if there exists a constant C such that

$$\sup_{0 \leqslant n \leqslant N} |\mathbb{E}p(X_n) - \mathbb{E}p(X(t_n))| \leqslant C(\varDelta t)^r,$$

with $N = T/\varDelta t$ and all sufficiently smooth function p. Consider the following one dimensional SDE

$$\begin{aligned} dX(t) &= \lambda X(t)dt + \mu X(t)dW(t), \quad 0 \leqslant t \leqslant T, \\ X(0) &= X_0, \end{aligned} \tag{5.2}$$

with $\lambda = 2$, $\mu = 1$, and $e^{\mathrm{w}}_{\Delta t} := |\mathbb{E}(X_N) - \mathbb{E}(X(T))|$ and verify numerically that the Euler-Maruyama method satisfies $e^{\mathrm{w}}_{\Delta t} \leqslant C\varDelta t$.

Solution

The plot is given in Figure 3.

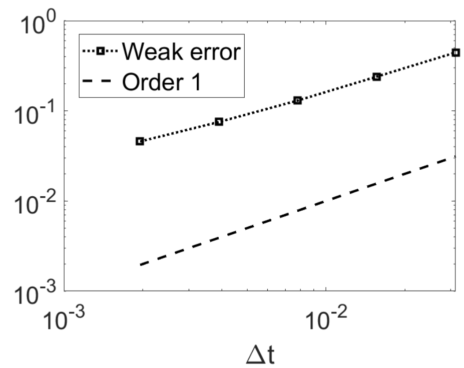

Figure 3: Weak order of convergence 1 for the EM method.