# Series 5 - October 16, 2024

## Exercise 1.

Let $g\colon [0,T] \to \mathbb{R}$ be a continuous function and let $b > 0$ and $X_0 \in \mathrm{L}^2(\Omega)$. Compute the solution of the following SDE for $t \in [0,T]$

$$\begin{aligned} \mathrm{d}X(t) &= -bX(t)\mathrm{d}t + g(t)\mathrm{d}W(t), \\ X(0) &= X_0. \end{aligned}$$

## Exercise 2.

Let $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$ and solve the $n$-dimensional SDE for $t \in [0,T]$

$$\mathrm{d}X(t) = AX(t)\mathrm{d}t + B\mathrm{d}W(t), \tag{2.1}$$

with initial condition $X(0) = X_0$ and where $W$ is an $m$-dimensional Brownian motion.
*Hint.* Generalize the one-dimensional case presented in the previous exercises.

## Exercise 3.

For $X_0 = \begin{pmatrix} X_0^1 & X_0^2 \end{pmatrix}^\top \in (\mathrm{L}^2(\Omega))^2$, solve the system of SDEs for $t \in [0,T]$

$$\begin{aligned} \mathrm{d}X_1(t) &= X_2(t)\mathrm{d}t + \mathrm{d}W_1(t), \\ \mathrm{d}X_2(t) &= X_1(t)\mathrm{d}t + \mathrm{d}W_2(t), \end{aligned} \tag{3.1}$$

with initial condition $X(0) = X_0$ and where $W_1$ and $W_2$ are two independent one-dimensional Brownian motions.
*Hint.* Use Exercise 1 or consider the processes

$$\begin{aligned} Y_1(t) &= X_1(t) + X_2(t), \\ Y_2(t) &= X_1(t) - X_2(t). \end{aligned} \tag{3.2}$$

## Exercise 4.

We recall that for random vectors $X \in \mathbb{R}^k$, $Y \in \mathbb{R}^m$ that are jointly Gaussian

$$\begin{pmatrix} X \\ Y \end{pmatrix} = \mathcal{N}\left( \begin{pmatrix} \mu_X \\ \mu_Y \end{pmatrix}, \begin{pmatrix} C_X & C_{XY} \\ C_{YX} & C_Y \end{pmatrix} \right), \tag{4.1}$$

the conditional distribution of $X$ given $Y$ is also Gaussian with the following distribution

$$X|Y \sim \mathcal{N}(\mu_X + C_{XY}C_Y^{-1}(Y-\mu_Y), C_X - C_{XY}C_Y^{-1}C_{YX}) \tag{4.2}$$

1) Let $\{B_t\}_{t\in[0,T]}$ be a Brownian motion on $(\Omega, \mathcal{F}, P, (\mathcal{F})_{t\geqslant 0})$. What is the conditional distribution of $B$ given $B_1 = 0$? The Gaussian process with such a distribution is called a Browian bridge.

2) Consider the stochastic process $\hat{B}_t = B_t - tB_1$. Show that $\hat{B}_t$ is a Gaussian process and has the distribution of a Brownian bridge.

3) Define the process $W_t$ as $dW_t = dB_t - \frac{B_1 - B_t}{1-t} dt$. Show that $W_t$ is a Brownian motion with respect to the filtration $\tilde{\mathcal{F}}_t = \sigma(B_u,\ u \leqslant t) \cup \sigma(B_1)$

   *Hint.* To show that $W_t - W_s$ is independent of $\tilde{\mathcal{F}}_s$ it is enough to show that $\mathbb{E}[(W_t - W_s)B_u] = 0$ for all $u \leqslant s$ and $u = 1$ since all variables are Gaussian (hence uncorrelation is equivalent to independence).

4) Consider now the SDE

$$d\xi_t = -\frac{\xi_t}{1-t} dt + dW_t, \tag{4.3}$$

   $\xi_0 = 0$ with $W_t$ as in the previous point. Solve explicitly this SDE. Show that $\{\xi_t\}_{t\in[0,1]}$ is a Gaussian process and has the distribution of a Brownian bridge. What is the relation between $\{\xi_t\}_{t\in[0,1]}$ and the process $\hat{B}_t = B_t - tB_1$ of point 2) ?

   *Hint.* Write the stochastic differential of $\hat{B}_t$ and replace $dB_t$ with $dW_t$.

# Exercise 5.

Consider the SDE

$$\begin{aligned} d\xi_t &= b(\xi_t, t)dt + \sigma(\xi_t, t)dB_t \\ \xi_0 &= x \end{aligned} \tag{5.1}$$

with the standard assumptions on $b : \mathbb{R}^d \times \mathbb{R}_+ \to \mathbb{R}^d$, $\sigma : \mathbb{R}^d \times \mathbb{R}_+ \to \mathbb{R}^{d\times m}$ (global Lipschitz and linear growth bound with constant $M$).

1) Suppose that $\sigma$ is bounded, i.e. $\exists k > 0$ such that $\|\sigma(x,t)\|_F \leqslant k$, for all $x, t$. Show that for any $T > 0$ there exists $c_T > 0$ such that for $R$ large enough

$$P(\sup_{0\leqslant t\leqslant T} |\xi_t| > R) \leqslant e^{-c_T R^2}, \tag{5.2}$$

   i.e. the process has Gaussian tails.

   *Hint.* use $L^1$ type bounds on $|\xi_t|$ on the set $A = \left\{ \sup_{0\leqslant t\leqslant T} \left|\int_0^t \sigma(\xi_s, s)dB_s\right| \geqslant \rho \right\}$ and the following exponential martingale inequality to bound $P(A)$: let $G \in M^2([0,T], \mathbb{R}^{d\times m})$ be s.t.

$$\int_0^T \theta^\top G_s G_s^\top \theta ds \leqslant c|\theta|^2$$

   for all $\theta \in \mathbb{R}^d$, and $I_t = \int_0^t G_s dB_s$. Then

$$\mathrm{P}\left( \sup_{0\leqslant t\leqslant T} |I_t| \geqslant \rho \right) \leqslant 2d\mathrm{e}^{-\frac{\rho^2}{2cd}}.$$

2) Show that $\sup_{0\leqslant t\leqslant T} |\xi_t|$ has all exponential moments finite i.e. $\mathbb{E}[\exp\{\lambda \sup_{0\leqslant t\leqslant T} |\xi_t|\}] < \infty$ for every $\lambda > 0$. Moreover, there exists $c^* > 0$ such that

$$\mathrm{E}\left[\exp\{c \sup_{0\leqslant t\leqslant T} |\xi_t|^2\}\right] < +\infty$$

   for all $c < c^*$.

   *Hint.* Use that for a positive random variable $x$ with $g \in C^1(\mathbb{R}_+, \mathbb{R})$ we have $\mathbb{E}[g(x)] = g(0) + \int_0^\infty g'(s)P(x \geqslant s)ds$

3) Remove the assumption of boundedness for $\sigma$. Show that for every $T > 0$, there exists a constant $c = c_T > 0$ such that for $R$ large enough

$$\mathrm{P}\left( \sup_{0\leqslant t\leqslant T} |\xi_t| > R \right) \leqslant \frac{1}{R^{c\log R}}.$$

   *Hint.* Write the stochastic differential of $Y_t = u(\xi_t)$ for $u(x) = \log(1 + |x|^2)$ and observe that $b^\top(\xi_t, t)\nabla u(\xi_t)$, $\sigma(\xi_t, t)\sigma^\top(\xi_t, t) : \nabla^2 u(\xi_t)$, $\sigma_{:,k}(\xi_t, t) \cdot \nabla u(\xi_t)$ $k = 1, \dots, m$ are all bounded functions.