Series 5 - October 16, 2024

Exercise 1.

Let $g\colon [0,T] \to \mathbb{R}$ be a continuous function and let $b > 0$ and $X_0 \in \mathrm{L}^2(\Omega)$. Compute the solution of the following SDE for $t \in [0,T]$

$$\begin{aligned} \mathrm{d}X(t) &= -bX(t)\mathrm{d}t + g(t)\mathrm{d}W(t), \\ X(0) &= X_0. \end{aligned}$$

Solution

We apply the variation of constants method. First, we find the solution of the homogeneous equation, which is $X(t) = P(t)X_0$ and then we look for a particular solution $Y(t)$. We consider the integrating factor

$$P^{-1}(t) = e^{-\left(\int_0^t -b\mathrm{d}s\right)} = e^{bt}$$

and considering a particular solution of the form

$$Y(t) = P^{-1}(t)X(t).$$

Then

$$\begin{aligned} \mathrm{d}Y_t =& [P^{-1}(t)]'X(t)\mathrm{d}t + P^{-1}(t)\mathrm{d}X(t) \\ &= be^{bt}X(t)\mathrm{d}t + e^{bt}(-bX(t)\mathrm{d}t + g(t)\mathrm{d}W(t)) \\ &= e^{bt}g(t)\mathrm{d}W(t). \end{aligned}$$

Hence

$$X(t) = X_0e^{-bt} + \int_0^t e^{-b(t-s)}g(s)\mathrm{d}W(s).$$

Exercise 2.

Let $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$ and solve the n-dimensional SDE for $t \in [0,T]$

$$\mathrm{d}X(t) = AX(t)\mathrm{d}t + B\mathrm{d}W(t), \tag{2.1}$$

with initial condition $X(0) = X_0$ and where W is an m-dimensional Brownian motion.
Hint. Generalize the one-dimensional case presented in the previous exercises.

Solution

We generalize the one-dimensional solution as

$$X(t) = e^{At}X_0 + \int_0^t e^{A(t-s)}B\mathrm{d}W(s).$$

We now prove using the Itô formula that it is indeed a solution. We have

$$Y(t) = e^{-At}X(t) = X_0 + \int_0^t e^{-As}B\mathrm{d}W(s),$$

hence the process $(Y(t), 0 \leqslant t \leqslant T)$ has the differential

$$\mathrm{d}Y(t) = e^{-At} B \mathrm{d}W(t).$$

Let us now consider the function $u\colon \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ defined as

$$u(t, x) = e^{At} x,$$

so that $X(t) = u(t, Y(t))$. We can apply the multidimensional Itô formula to each of the components of $u(t, Y(t))$. The partial derivatives are given by

$$\begin{aligned}
\partial_t (u(t,x)_i) &= \sum_{k,l=1}^{n} A_{ik} (e^{At})_{kl} x_l, && i = 1, \dots, n, \\
\partial_{x_j} (u(t,x)_i) &= (e^{At})_{ij}, && i, j = 1, \dots, n, \\
\partial_{x_j x_k} (u(t,x)_i) &= 0, && i, j, k = 1, \dots, n.
\end{aligned}$$

Hence, we have

$$\begin{aligned}
(\mathrm{d}X(t))_i = \mathrm{d}(u(t, Y(t)))_i &= \sum_{k,l=1}^{n} A_{ik} (e^{At})_{kl} Y(t)_l \mathrm{d}t + \sum_{j=1}^{n} (e^{At})_{ij} \mathrm{d}Y(t)_j \\
&= \sum_{k,l=1}^{n} A_{ik} (e^{At})_{kl} Y(t)_l \mathrm{d}t + \sum_{j,k=1}^{n} \sum_{l=1}^{m} (e^{At})_{ij} (e^{-At})_{jk} B_{kl} \mathrm{d}W(t)_l \\
&= (A e^{At} e^{-At} X(t))_i \mathrm{d}t + (e^{At} e^{-At} B \mathrm{d}W(t))_i \\
&= (AX(t))_i \mathrm{d}t + (B \mathrm{d}W(t))_i,
\end{aligned}$$

which proves that $X(t)$ is a solution.

Exercise 3.

For $X_0 = \begin{pmatrix} X_0^1 & X_0^2 \end{pmatrix}^\top \in (\mathrm{L}^2(\Omega))^2$, solve the system of SDEs for $t \in [0, T]$

$$\begin{aligned}
\mathrm{d}X_1(t) &= X_2(t) \mathrm{d}t + \mathrm{d}W_1(t), \\
\mathrm{d}X_2(t) &= X_1(t) \mathrm{d}t + \mathrm{d}W_2(t),
\end{aligned} \tag{3.1}$$

with initial condition $X(0) = X_0$ and where W_1 and W_2 are two independent one-dimensional Brownian motions.
Hint. Use Exercise 1 or consider the processes

$$\begin{aligned}
Y_1(t) &= X_1(t) + X_2(t), \\
Y_2(t) &= X_1(t) - X_2(t).
\end{aligned} \tag{3.2}$$

Solution

We consider the previous exercise with $n = m = 2$ and A, B given by

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

We have now to compute $\exp(At)$. This can be done by diagonalizing A on an orthonormal basis as

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \frac{\sqrt{2}}{2} \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \frac{\sqrt{2}}{2} \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}^\top = U \Lambda U^\top,$$

which yields through $\exp(At) = U\exp(\Lambda t)U^\top$

$$e^{At} = \frac{1}{2}\begin{pmatrix} e^t + e^{-t} & e^t - e^{-t} \\ e^t - e^{-t} & e^t + e^{-t} \end{pmatrix}.$$

Otherwise, one can compute

$$\exp(At) = \sum_{j=0}^{\infty} \frac{(At)^j}{j!}.$$

Noticing that

$$(At)^j = t^j \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \text{if } j \text{ even},$$

$$(At)^j = t^j \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \text{if } j \text{ odd},$$

we have

$$\begin{aligned} e^{At} &= \sum_{j=0}^{\infty} \frac{1}{(2j)!}\begin{pmatrix} t^{2j} & 0 \\ 0 & t^{2j} \end{pmatrix} + \sum_{j=0}^{\infty} \frac{1}{(2j+1)!}\begin{pmatrix} 0 & t^{2j+1} \\ t^{2j+1} & 0 \end{pmatrix} \\ &= \begin{pmatrix} \cosh t & 0 \\ 0 & \cosh t \end{pmatrix} + \begin{pmatrix} 0 & \sinh t \\ \sinh t & 0 \end{pmatrix} \\ &= \frac{1}{2}\begin{pmatrix} e^t + e^{-t} & e^t - e^{-t} \\ e^t - e^{-t} & e^t + e^{-t} \end{pmatrix}. \end{aligned}$$

Plugging into the solution of the general equation the value of $\exp(At)$, we obtain

$$\begin{pmatrix} X_1(t) \\ X_2(t) \end{pmatrix} = \frac{1}{2}\begin{pmatrix} (e^t + e^{-t})X_0^1 + (e^t - e^{-t})X_0^2 \\ (e^t - e^{-t})X_0^1 + (e^t + e^{-t})X_0^2 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} \int_0^t (e^{t-s} + e^{s-t})\mathrm{d}W_1(s) \\ \int_0^t (e^{t-s} + e^{s-t})\mathrm{d}W_2(s) \end{pmatrix}. \tag{3.3}$$

Alternatively, the processes $Y_1 = X_1 + X_2$ and $Y_2 = X_1 - X_2$ satisfy

$$\begin{aligned} \mathrm{d}Y_1(t) &= Y_1\mathrm{d}t + \sqrt{2}\mathrm{d}\widetilde{W}_1(t), \qquad Y_1(0) = X_1(0) + X_2(0), \\ \mathrm{d}Y_2(t) &= -Y_2\mathrm{d}t + \sqrt{2}\mathrm{d}\widetilde{W}_2(t), \quad Y_2(0) = X_1(0) - X_2(0). \end{aligned}$$

where $\widetilde{W}_1 = (W_1 + W_2)/\sqrt{2}$ and $\widetilde{W}_2 = (W_1 - W_2)/\sqrt{2}$. We have by Series 8 Ex. 3 (or 6)

$$\begin{aligned} Y_1(t) &= Y_1(0) + \int_0^t e^{t-s}\sqrt{2}\mathrm{d}\widetilde{W}_1(s), \\ Y_2(t) &= Y_2(0) + \int_0^t e^{-(t-s)}\sqrt{2}\mathrm{d}\widetilde{W}_2(t). \end{aligned}$$

Computing $X_1 = (Y_1 + Y_2)/2$ and $X_2 = (Y_1 - Y_2)/2$ one obtains the same result as (3.3).

Exercise 4.

We recall that for random vectors $X \in \mathbb{R}^k$, $Y \in \mathbb{R}^m$ that are jointly Gaussian

$$\begin{pmatrix} X \\ Y \end{pmatrix} = \mathcal{N}\left(\begin{pmatrix} \mu_X \\ \mu_Y \end{pmatrix}, \begin{pmatrix} C_X & C_{XY} \\ C_{YX} & C_Y \end{pmatrix}\right), \tag{4.1}$$

the conditional distribution of X given Y is also Gaussian with the following distribution

$$X|Y \sim \mathcal{N}(\mu_X + C_{XY}C_Y^{-1}(Y - \mu_Y), C_X - C_{XY}C_Y^{-1}C_{YX}) \tag{4.2}$$

1) Let $\{B_t\}_{t\in[0,T]}$ be a Brownian motion on $(\Omega, \mathcal{F}, P, (\mathcal{F})_{t\geqslant 0})$. What is the conditional distribution of B given $B_1 = 0$? The Gaussian process with such a distribution is called a Browian bridge.

2) Consider the stochastic process $\hat{B}_t = B_t - tB_1$. Show that $\hat{B}_t$ is a Gaussian process and has the distribution of a Brownian bridge.

3) Define the process W_t as $dW_t = dB_t - \frac{B_1 - B_t}{1-t} dt$. Show that W_t is a Brownian motion with respect to the filtration $\tilde{\mathcal{F}}_t = \sigma(B_u,\ u \leqslant t) \cup \sigma(B_1)$

   Hint. To show that $W_t - W_s$ is independent of $\tilde{\mathcal{F}}_s$ it is enough to show that $\mathbb{E}[(W_t - W_s)B_u] = 0$ for all $u \leqslant s$ and $u = 1$ since all variables are Gaussian (hence uncorrelation is equivalent to independence).

4) Consider now the SDE
$$d\xi_t = -\frac{\xi_t}{1-t} dt + dW_t, \tag{4.3}$$
   $\xi_0 = 0$ with W_t as in the previous point. Solve explicitly this SDE. Show that $\{\xi_t\}_{t \in [0,1]}$ is a Gaussian process and has the distribution of a Brownian bridge. What is the relation between $\{\xi_t\}_{t \in [0,1]}$ and the process $\hat{B}_t = B_t - tB_1$ of point 2) ?

   Hint. Write the stochastic differential of $\hat{B}_t$ and replace dB_t with dW_t.

Solution

1) We apply formula (4.2) to $X = \left(B_{t_1}, \dots, B_{t_m}\right), Y = B_1$. We have $C_Y = 1$ whereas, since $\mathrm{Cov}\left(B_{t_i}, B_1\right) = t_i \wedge 1 = t_i$,
$$C_{X,Y} = \begin{pmatrix} t_1 \\ \vdots \\ t_m \end{pmatrix}$$
   and
$$C_{X,Y} C_Y^{-1} C_{X,Y}^T = \begin{pmatrix} t_1 \\ \vdots \\ t_m \end{pmatrix} (t_1 \dots t_m)$$
   the matrix $C_{X,Y} C_Y^{-1} C_{X,Y}^T$ has $t_i t_j$ as its (i,j)-th entry. As C_X is the matrix $\left(t_i \wedge t_j\right)_{i,j}$, the conditional law is Gaussian with covariance matrix $C_X - C_{X,Y} C_Y^{-1} C_{X,Y} = \left(t_i \wedge t_j - t_i t_j\right)_{i,j}$, i.e.
$$C := \begin{pmatrix} t_1(1-t_1) & t_1(1-t_2) & \dots t_1(1-t_m) \\ t_1(1-t_2) & t_2(1-t_2) & \dots t_2(1-t_m) \\ & \ddots & \\ t_1(1-t_m) & t_2(1-t_m) & \dots t_m(1-t_m) \end{pmatrix}$$
   and mean (here $\mathrm{E}[X] = 0, \mathrm{E}[Y] = 0$)
$$C_{X,Y} C_Y^{-1} Y = \begin{pmatrix} t_1 \\ \vdots \\ t_m \end{pmatrix} B_1 \tag{4.4}$$
   Let us denote $\{\hat{B}_t\}_{t \in [0,T]}$ the conditional process. The previous calculation shows that all finite dimensional distributions are Gaussian and $\hat{X} = \left(\hat{B}_{t_1}, \dots, \hat{B}_{t_m}\right) = X | Y = 0 \sim \mathcal{N}(0, C)$ with C defined in (1). Hence the process $\hat{B}$ is Gaussian with mean $\mu_{\hat{B}} = 0$ and covariance $\mathrm{Cov}_{\hat{B}}(s,t) = s \wedge t - st$.

2) Let $0 \leqslant t_1 < \dots < t_m \leqslant 1$. Clearly, $\hat{B}_t$ is a Gaussian process (all finite dimensional distributions are Gaussian). A straightforward calculation $\mathbb{E}[\hat{B}_t] = 0$ and $\mathrm{Cov}_{\hat{B}}(s,t) = \mathbb{E}[(B_s - sB_1)(B_t - tB_1)] = s \wedge t - st$, which coincide with the mean and the covariance of a Brownian bridge.¨

3) For the first property of the Brownian motion (independent increments), as the r.v.'s $W_t - W_s, B_1, B_v, \underset{\sim}{z}_v, v \leqslant s$ form a Gaussian family, it suffices to show that $W_t - W_s$ is orthogonal to $B_v, 0 \leqslant v \leqslant s$, and to B_1. We have

$$\mathrm{E}[(W_t - W_s)B_1] = \mathrm{E}[(B_t - B_s)B_1] - \int_s^t \frac{\mathrm{E}[(B_1 - B_u)B_1]}{1-u} du = t - s - \int_s^t du = 0$$

and, for $v \leqslant s$,

$$\mathrm{E}[(W_t - W_s)B_v] = \mathrm{E}[(B_t - B_s)B_v] - \int_s^t \frac{\mathrm{E}[(B_1 - B_u)B_v]}{1-u} du = 0$$

For the second property (Gaussian increments) clearly $\{W_t\}$ is a Gaussian process so we just need to check that it has the covariance of a Brownian motion. Let us prove that $\mathrm{E}[W_t W_s] = s$, for $0 \leqslant s \leqslant t$. This is elementary, albeit laborious. If $s < t \leqslant 1$, note that $\mathrm{E}[W_t W_s] = \mathrm{E}[(W_t - W_s)W_s] + \mathrm{E}[W_s^2] = \mathrm{E}[W_s^2]$, thanks to the previous property. Hence we are reduced to the computation of

$$\begin{aligned}
\mathrm{E}[W_s^2] &= \mathrm{E}(B_s^2) - 2\int_0^s \mathrm{E}\Big[B_s \frac{B_1 - B_u}{1-u}\Big] du + \int_0^s dv \int_0^s \mathrm{E}\Big[\frac{B_1 - B_v}{1-v}\frac{B_1 - B_u}{1-u}\Big] du \\
&= s - 2\int_0^s \frac{s-u}{1-u} du + \int_0^s dv \int_0^s \frac{1-u-v+u\wedge v}{(1-v)(1-u)} du \\
&= s - 2I_1 + I_2
\end{aligned}$$

With patience one can compute I_2 and find that it is equal to $2I_1$, which gives the result. The simplest way to check that $I_2 = 2I_1$ is to observe that the integrand in I_2 is a function of (u, v) that is symmetric in u, v. Hence

$$\begin{aligned}
I_2 &= \int_0^s dv \int_0^s \frac{1-u-v+u\wedge v}{(1-v)(1-u)} du = 2 \iint_{v \leqslant u} \frac{1-u-v+u\wedge v}{(1-v)(1-u)} du dv \\
&= 2\int_0^s dv \int_v^s \frac{1-u}{(1-v)(1-u)} du = 2\int_0^s \frac{s-v}{1-v} dv = 2I_1
\end{aligned}$$

Hence $\mathrm{E}[W_t W_s] = s \wedge t$. If $s \leqslant t$, the r.v. $W_t - W_s$ is centered Gaussian as $(W_t)_t$ is clearly a centered Gaussian process, which together with the first property, completes the proof that W is a $\left(\hat{\mathscr{F}}_t\right)_t$-Brownian motion.

Finally, we have

$$dB_t = A_t dt + dW_t$$

with

$$A_t = \frac{B_1 - B_t}{1-t}.$$

Hence, since A is adapted to $\left(\hat{\mathscr{F}}_t\right)_t$, B is an Ito process with respect to the new Brownian motion W.

4) Let us follow the idea of the variation of constants applied to a general SDE. More precisely, we formally consider a general SDE

$$\begin{aligned}
x'_t &= b(t)x_t + \sigma(t)dB_t \\
x_0 &= x
\end{aligned} \tag{4.5}$$

as an ordinary differential equation where the noise is considered as a external force term. The solution of the ordinary differential equation without noise

$$\begin{aligned}
x'_t &= b(t)x_t \\
x_0 &= x
\end{aligned}$$

is $x_t = \mathrm{e}^{\Lambda(t)}x$, where $\Lambda(t) = \int_0^t b(s)ds$. Let us look for a solution of the form $x_t = \mathrm{e}^{\Lambda(t)}C(t)$ for (4.5). One sees easily that C must be the solution of

$$\mathrm{e}^{\Lambda(t)}dC(t) = \sigma(t)dB_t$$

i.e.

$$C(t) = \int_0^t \mathrm{e}^{-\Lambda(s)}\sigma(s)dB_s$$

The solution is therefore

$$\xi_t = \mathrm{e}^{\Lambda(t)}\xi_0 + \mathrm{e}^{\Lambda(t)}\int_0^t \mathrm{e}^{-\Lambda(s)}\sigma(s)dW_s$$

As the stochastic integral of a deterministic function is, as a function of the integration endpoint, a Gaussian process, ξ is Gaussian. Obviously $\mathrm{E}(\xi_t) = \mathrm{e}^{\Lambda(t)}\xi_0$. Let us compute its covariance function. Let $s \leqslant t$ and let

$$Y_t = \mathrm{e}^{\Lambda(t)}\int_0^t \mathrm{e}^{-\Lambda(s)}\sigma(s)dB_s$$

Then,

$$\begin{aligned}
\mathrm{Cov}(\xi_t, \xi_s) &= \mathrm{E}(Y_t Y_s^*) \\
&= \mathrm{E}\Big[\mathrm{e}^{\Lambda(t)}\int_0^t \mathrm{e}^{-\Lambda(u)}\sigma(u)dB_u \Big(\mathrm{e}^{\Lambda(s)}\int_0^s \mathrm{e}^{-\Lambda(v)}\sigma(v)dB_v\Big)^*\Big] \\
&= \mathrm{e}^{\Lambda(t)}\mathrm{E}\Big[\int_0^t \mathrm{e}^{-\Lambda(u)}\sigma(u)dB_u \Big(\int_0^s \mathrm{e}^{-\Lambda(v)}\sigma(v)dB_v\Big)^*\Big]\mathrm{e}^{\Lambda(s)^*} \\
&= \mathrm{e}^{\Lambda(t)}\int_0^s \mathrm{e}^{-\Lambda(u)}\sigma(u)\sigma^*(u)\mathrm{e}^{-\Lambda(u)^*}du\mathrm{e}^{\Lambda(s)^*}
\end{aligned}$$

In particular, if $m = 1$ the covariance is

$$\mathrm{e}^{\Lambda(t)}\mathrm{e}^{\Lambda(s)}\int_0^s \mathrm{e}^{-2\Lambda(u)}\sigma^2(u)du$$

Therefore, for (4.3) we have

$$\Lambda(t) = \int_0^t -\frac{1}{1-s}ds = \log(1-t)$$

Therefore $\mathrm{e}^{\Lambda(t)} = 1 - t$ and the solution of (9.47) is

$$\xi_t = (1-t)\xi_0 + (1-t)\int_0^t \frac{dW_s}{1-s}$$

Hence $\mathrm{E}[\xi_t] = (1-t)\xi_0$. ξ is a Gaussian process with covariance function

$$K(t,s) = (1-t)(1-s)\int_0^s \frac{1}{(1-u)^2}du = (1-t)(1-s)\Big(\frac{1}{1-s} - 1\Big) = s(1-t)$$

for $s \leqslant t$. If $x = 0$, then $\mathrm{E}(\xi_t) = 0$ for every t and the process has the same mean and covariance functions as a Brownian bridge.

Finally, one has

$$\begin{aligned}
d\widehat{B}_t &= dB_t - B_1 \\
&= dW_t + \frac{B_1 - B_t}{1-t}dt - B_1 \\
&= \frac{tB_1 - B_t}{1-t}dt + dW_t \\
&= -\frac{\widehat{B}_t}{1-t}dt + dW_t
\end{aligned} \tag{4.6}$$

Thus, $\widehat{B}$ solves (4.3).

Exercise 5.

Consider the SDE

$$\begin{aligned} d\xi_t &= b(\xi_t,t)dt + \sigma(\xi_t,t)dB_t \\ \xi_0 &= x \end{aligned} \tag{5.1}$$

with the standard assumptions on $b : \mathbb{R}^d \times \mathbb{R}_+ \to \mathbb{R}^d$, $\sigma : \mathbb{R}^d \times \mathbb{R}_+ \to \mathbb{R}^{d\times m}$ (global Lipschitz and linear growth bound with constant M).

1) Suppose that σ is bounded, i.e. $\exists k > 0$ such that $\|\sigma(x,t)\|_F \leqslant k$, for all x, t. Show that for any $T > 0$ there exists $c_T > 0$ such that for R large enough

$$P(\sup_{0\leqslant t\leqslant T} |\xi_t| > R) \leqslant e^{-c_T R^2}, \tag{5.2}$$

i.e. the process has Gaussian tails.

Hint. use L^1 type bounds on $|\xi_t|$ on the set $A = \left\{\sup_{0\leqslant t\leqslant T}\left|\int_0^t \sigma(\xi_s,s)dB_s\right| \geqslant \rho\right\}$ and the following exponential martingale inequality to bound $P(A)$: let $G \in M^2([0,T],\mathbb{R}^{d\times m})$ be s.t.

$$\int_0^T \theta^\top G_s G_s^\top \theta ds \leqslant c|\theta|^2$$

for all $\theta \in \mathbb{R}^d$, and $I_t = \int_0^t G_s dB_s$. Then

$$\mathrm{P}\left(\sup_{0\leqslant t\leqslant T} |I_t| \geqslant \rho\right) \leqslant 2d\mathrm{e}^{-\frac{\rho^2}{2cd}}.$$

2) Show that $\sup_{0\leqslant t\leqslant T} |\xi_t|$ has all exponential moments finite i.e. $\mathbb{E}[\exp\{\lambda \sup_{0\leqslant t\leqslant T} |\xi_t|\}] < \infty$ for every $\lambda > 0$. Moreover, there exists $c^* > 0$ such that

$$\mathrm{E}\left[\exp\{c \sup_{0\leqslant t\leqslant T} |\xi_t|^2\}\right] < +\infty$$

for all $c < c^*$.

Hint. Use that for a positive random variable x with $g \in C^1(\mathbb{R}_+,\mathbb{R})$ we have $\mathbb{E}[g(x)] = g(0) + \int_0^\infty g'(s)P(x \geqslant s)ds$

3) Remove the assumption of boundedness for σ. Show that for every $T > 0$, there exists a constant $c = c_T > 0$ such that for R large enough

$$\mathrm{P}\left(\sup_{0\leqslant t\leqslant T} |\xi_t| > R\right) \leqslant \frac{1}{R^{c\log R}}.$$

Hint. Write the stochastic differential of $Y_t = u(\xi_t)$ for $u(x) = \log(1 + |x|^2)$ and observe that $b^\top(\xi_t,t)\nabla u(\xi_t)$, $\sigma(\xi_t,t)\sigma^\top(\xi_t,t) : \nabla^2 u(\xi_t)$, $\sigma_{:,k}(\xi_t,t)\cdot\nabla u(\xi_t)$ $k = 1,\dots,m$ are all bounded functions.

Solution

1) Via the exponential martingale inequality we have

$$\mathrm{P}\left(\sup_{0\leqslant t\leqslant T}\left|\int_0^t \sigma(\xi_s,s)dB_s\right| \geqslant \rho\right) \leqslant 2d\mathrm{e}^{-c_0\rho^2}$$

where $c_0 = (2Tdk)^{-1}$. If we define $A = \left\{\sup_{0\leqslant t\leqslant T}\left|\int_0^t \sigma(\xi_s,s)dB_s\right| < \rho\right\}$, then on A we have, for $t \leqslant T$,

$$|\xi_t| \leqslant |x| + M\int_0^t (1 + |\xi_s|)ds + \rho$$

i.e.

$$\xi_T^* := \sup_{0\leqslant t\leqslant T} |\xi_t| \leqslant (|x| + MT + \rho) + M\int_0^T \xi_s^* ds$$

and by Gronwall's Lemma $\xi_T^* \leqslant (|x| + MT + \rho)\mathrm{e}^{MT}$. Therefore, if $|x| \leqslant K$,

$$\mathrm{P}\Big(\sup_{0\leqslant t\leqslant T} |\xi_t| > (K + MT + \rho)\mathrm{e}^{MT}\Big) \leqslant \mathrm{P}\Big(\sup_{0\leqslant t\leqslant T}\Big|\int_0^t \sigma(\xi_s, s) dB_s\Big| \geqslant \rho\Big) \leqslant 2d\mathrm{e}^{-c_0\rho^2}$$

Setting $R = (K + MT + \rho)\mathrm{e}^{MT}$ we have $\rho = R\mathrm{e}^{-MT} - (|k| + MT)$ and therefore

$$\mathrm{P}\Big(\sup_{0\leqslant t\leqslant T} |\xi_t| > R\Big) \leqslant 2d \exp\Big(-c_0\big(R\mathrm{e}^{-MT} - (K + MT)\big)^2\Big)$$

from which we obtain that, for every constant $c = c_T$ strictly smaller than

$$c_0\mathrm{e}^{-2MT} = \frac{\mathrm{e}^{-2MT}}{2Tmk}$$

the inequality (5.2) holds for R large enough.

2) Using the hint with $g(x) = e^{\lambda x}$ and the estimate from the previous point we have

$$\mathrm{E}\left[\mathrm{e}^{\lambda\left(\sup\limits_{0\leqslant t\leqslant T} |\xi_t|\right)}\right] = 1 + \int_0^{+\infty} \lambda\mathrm{e}^{\lambda s}\mathrm{P}\Big(\sup_{0\leqslant t\leqslant T} |\xi_t| \geqslant s\Big)dt < +\infty, \quad \lambda \in \mathbb{R}.$$

If we take instead $g(x) = e^{\alpha x^2}$, then

$$\mathrm{E}\left[\mathrm{e}^{\alpha\left(\sup\limits_{0\leqslant t\leqslant T} |\xi_t|^2\right)}\right] = 1 + \int_0^{+\infty} \alpha s\mathrm{e}^{\alpha s^2}\mathrm{P}\Big(\sup_{0\leqslant t\leqslant T} |\xi_t|^2 \geqslant s\Big)dt < +\infty, \quad \alpha \in \mathbb{R}.$$

which is bounded for all $\alpha < C_T$.

3) If $u(x) = \log(1 + |x|^2)$, let us compute the derivatives:

$$u_{x_i}(x) = \frac{2x_i}{1 + |x|^2}$$
$$u_{x_ix_j}(x) = \frac{2\delta_{ij}}{1 + |x|^2} - \frac{4x_ix_j}{(1 + |x|^2)^2}$$

In particular, as $|x| \to +\infty$ the first-order derivatives go to 0 at least as $x \mapsto |x|^{-1}$ and the second-order derivatives at least as $x \mapsto |x|^{-2}$. By Ito's formula the process $Y_t = \log\big(1 + |\xi_t|^2\big)$ has stochastic differential

$$dY_t = \Big(\sum_{i=1}^m u_{x_i}(\xi_t)b_i(\xi_t, t) + \frac{1}{2}\sum_{i,j}^m u_{x_ix_j}(\xi_t)a_{ij}(\xi_t, t)\Big)dt + \sum_{i=1}^m\sum_{j=1}^d u_{x_i}(\xi_t)\sigma_{ij}(\xi_t, t)dB_j(t)$$

where $a = \sigma\sigma^*$. As b and σ are assumed to satisfy the linear growth bound property, it is clear that all the terms $u_{x_i}b_i, u_{x_ix_j}a_{ij}, u_{x_i}\sigma_{ij}$ are bounded. We can therefore apply 1) which guarantees that there exists a constant $c > 0$ such that, for large ρ,

$$\mathrm{P}\Big(\sup_{0\leqslant t\leqslant T} \log\big(1 + |\xi_t|^2\big) \geqslant \log\rho\Big) \leqslant \mathrm{e}^{-c(\log\rho)^2}$$

i.e. $\mathrm{P}\big(\xi_T^* \geqslant \sqrt{\rho - 1}\big) \leqslant \mathrm{e}^{-c(\log\rho)^2}$. Letting $R = \sqrt{\rho - 1}$, i.e. $\rho = R^2 + 1$, the inequality becomes, for large R,

$$\mathrm{P}(\xi_T^* \geqslant R) \leqslant \mathrm{e}^{-c(\log(R^2+1))^2} \leqslant \mathrm{e}^{-c(\log R)^2} = \frac{1}{R^{c\log R}}.$$