

Series 4 - October 9, 2024

Exercise 1.

Let $G \in M^2(0, T)$ with $t \mapsto G(t, \omega)$ continuous for almost every ω and consider a partition $P_\Delta = \{0 = t_0 < t_1 < \dots < t_{m_\Delta} = T\}$ of $[0, T]$ of size Δ , i.e., $t_j = j\Delta$ for $j = 0, \dots, m_\Delta$. Assume that $\mathbb{E}[G(t)G(s)]$ is a continuous function of t and s . Show that

$$\lim_{\Delta \rightarrow 0} \sum_{j=1}^{m_\Delta} G(t_{j-1})(W(t_j) - W(t_{j-1})) = \int_0^T G(t) dW(t) \quad \text{in } L^2.$$

Solution

From G define the step process $G_m(r, \omega) = \sum_{j=1}^m G(t_{j-1}, \omega) \chi_{[t_{j-1}, t_j[}(r)$ and define

$$I_m := \mathbb{E} \left[\left(\sum_{j=1}^{m_\Delta} G(t_{j-1})(W(t_j) - W(t_{j-1})) - \int_0^T G(t) dW(t) \right)^2 \right]. \quad (1.1)$$

Then, by the Itô isometry we obtain

$$I_m = \mathbb{E} \left[\left(\int_0^T (G_m(t) - G(t)) dW(t) \right)^2 \right] = \int_0^T \mathbb{E}[(G_m(t) - G(t))^2] dt. \quad (1.2)$$

Due to the assumption we have

$$\lim_{s \rightarrow t} \mathbb{E}[(G(s) - G(t))^2] = 0, \quad (1.3)$$

which implies for all $t \in [0, T]$

$$\lim_{m \rightarrow \infty} \mathbb{E}[(G_m(t) - G(t))^2] = 0. \quad (1.4)$$

Moreover, we also have

$$\mathbb{E}[(G_m(t) - G(t))^2] \leq 2(\mathbb{E}[G_m(t)^2] + \mathbb{E}[G(t)^2]) \leq 4 \max_{t \in [0, T]} \mathbb{E}[G(t)^2], \quad (1.5)$$

where the right-hand side is finite by assumption. Therefore, applying the dominated convergence theorem we deduce that $I_m \rightarrow 0$ as $m \rightarrow \infty$, which concludes the proof.

Exercise 2.

Let $(W(t), t \geq 0)$ be a one-dimensional Brownian motion. Without implement the Itô formula but using the construction of the stochastic integral, show that:

$$i) \quad d(W^2) = 2WdW + dt,$$

$$ii) \quad d(tW) = Wdt + tdW.$$

Solution

i) We already know that

$$\int_0^T W(t)dW(t) = \frac{W(T)^2}{2} - \frac{T}{2},$$

and hence

$$\int_r^s W(t)dW(t) = \frac{W(s)^2 - W(r)^2}{2} - \frac{s-r}{2},$$

which gives

$$W(s)^2 = W(r)^2 + \int_r^s dt + \int_r^s 2W(t)dW(t),$$

which implies the result.

ii) Observe that for a sequence of partitions $P = \{r = t_0 < t_1 < \dots < t_m = s\}$ such that $\max_{1 \leq j \leq m} |t_j - t_{j-1}| \rightarrow 0$ we have

$$\int_r^s t dW(t) = \lim_{m \rightarrow \infty} \sum_{j=1}^m t_{j-1} (W(t_j) - W(t_{j-1})) \text{ in } L^2(\Omega).$$

As $t \mapsto W(t)$ is continuous a.s., we also have

$$\int_r^s W(t)dt = \lim_{m \rightarrow \infty} \sum_{j=1}^m W(t_j)(t_j - t_{j-1}),$$

indeed since it is an ordinary Riemann sum, the integrand can be evaluated at any point in $[t_{j-1}, t_j]$. Therefore, we obtain

$$\begin{aligned} \int_r^s W(t)dt + \int_r^s t dW &= \lim_{m \rightarrow \infty} \sum_{j=1}^m t_{j-1} (W(t_j) - W(t_{j-1})) + W(t_j)(t_j - t_{j-1}) \\ &= rW(r) - sW(s), \end{aligned}$$

which shows the result.

Exercise 3.

Let $F: [a, b] \times \mathbb{R} \rightarrow \mathbb{R}$ be a continuous function with continuous partial derivatives $\partial F / \partial t, \partial F / \partial x, \partial^2 F / \partial x^2$ and let $(\partial F / \partial x)(t, x) = f(t, x)$. Show the following analogue of the fundamental theorem of the Leibniz-Newton calculus for the Itô calculus

$$\int_a^b f(t, W(t))dW(t) = F(t, W(t)) \Big|_a^b - \int_a^b \left(\frac{\partial F}{\partial t}(t, W(t)) + \frac{1}{2} \frac{\partial^2 F}{\partial x^2}(t, W(t)) \right) dt.$$

In particular, when $F(t, x) = F(x)$ is independent of t , it reads

$$\int_a^b f(W(t))dW(t) = F(W(t)) \Big|_a^b - \frac{1}{2} \int_a^b f'(W(t))dt.$$

Then, use this formula to compute

$$\int_0^t W(s)e^{W(s)}dW(s).$$

Solution

The first result is obtained applying the Itô formula to compute $dF(t, W(t))$. Then, let $f(x) = xe^x$, which gives

$$\begin{aligned} F(x) &= \int_0^x ye^y dy = (x-1)e^x + 1, \\ f'(x) &= (x+1)e^x. \end{aligned}$$

The formula yields

$$\int_0^t W(s)e^{W(s)}dW(s) = (W(t) - 1)e^{W(t)} + 1 - \frac{1}{2} \int_0^t (1 + W(s))e^{W(s)}ds.$$

Exercise 4.

Compute $\mathbb{E}\left[B_s \int_0^t B_u dB_u\right]$ and $\mathbb{E}\left[B_s^2 \left(\int_0^t B_u dB_u\right)^2\right]$ for $0 \leq s \leq t$.

Recall: If $\xi \sim \mathcal{N}(0, 1)$, then $\mathbb{E}[\xi^4] = 3$.

Solution

Let $s \leq t$. We have $B_s = \int_0^s 1_{[0,s]}(v)dB_v$ therefore,

$$\mathbb{E}\left(B_s \int_0^t B_u dB_u\right) = \mathbb{E}\left(\int_0^s 1_{[0,s]}(v)dB_v \int_0^t B_u dB_u\right) = \int_0^s \mathbb{E}[1_{[0,s]}(u)B_u]du = 0$$

If $t \leq s$, the same argument leads to the same result. For the second moment, we have

$$\mathbb{E}\left[B_s^2 \left(\int_s^t B_u dB_u\right)^2\right] = \mathbb{E}\left[\left(\int_s^t B_s B_u dB_u\right)^2\right] = \mathbb{E}\left[\int_s^t B_s^2 B_u^2 du\right]$$

Recalling that $s \leq u$ so that $B_u - B_s$ is independent of B_s ,

$$\mathbb{E}(B_s^2 B_u^2) = \mathbb{E}[B_s^2 (B_u - B_s + B_s)^2] = \underbrace{\mathbb{E}[B_s^2 (B_u - B_s)^2]}_{=s(u-s)} + 2 \underbrace{\mathbb{E}[B_s^3 (B_u - B_s)]}_{=0} + \mathbb{E}(B_s^4).$$

We can write $B_s = \sqrt{s}Z$ where $Z \sim N(0, 1)$ and therefore $\mathbb{E}(B_s^4) = s^2 \mathbb{E}(Z^4) = 3s^2$. In conclusion, $\mathbb{E}(B_s^2 B_u^2) = s(u - s) + 3s^2$ and

$$\mathbb{E}\left[B_s^2 \left(\int_s^t B_u dB_u\right)^2\right] = \int_s^t s(u - s) + 3s^2 du = \frac{1}{2}s(t - s)^2 + 3s^2(t - s).$$

Exercise 5.

Let $X \in M^2([0, T])$ and consider the stochastic integral $I_t = \int_0^t X_s dB_s$.

1) Use Itô's formula to show that

$$|I_t|^p = p \int_0^t |I_s|^{p-1} \operatorname{sgn}(I_s) X_s dB_s + \frac{1}{2} p(p-1) \int_0^t |I_s|^{p-2} X_s^2 ds.$$

2) Assume $|I_t| \leq K$ a.s. for some $K > 0$. Deduce that there exists $c_p = C(T, p)$ such that

$$\mathbb{E}\left(\sup_{0 \leq t \leq T} \left|\int_0^t X_s dB_s\right|^p\right) \leq c_p \mathbb{E}\left[\left(\int_0^T |X_s|^2 ds\right)^{\frac{p}{2}}\right] \leq c_p T^{\frac{p-2}{2}} \mathbb{E}\left(\int_0^T |X_s|^p ds\right)$$

for some constant $c_p > 0$.

Hint. Use Doob's martingale and Holder inequalities.

3) Show that 2) works without asking for $|I_t| \leq K$ a.s. .

Hint. Take $\tau_n = \inf\{t \leq T; |I_t| \geq n\}$ and consider the martingale $I_{t \wedge \tau_n}$. Then take the limit for $n \rightarrow \infty$ and conclude by Fatou's lemma.

Solution

- 1) Let us apply Ito's formula to the function $f(x) = |x|^p$ (which is twice differentiable, as $p \geq 2$) and to the process I whose stochastic differential is $dI_t = X_t dB_t$. We have $f'(x) = p \operatorname{sgn}(x)|x|^{p-1}$, $f''(x) = p(p-1)|x|^{p-2}$, where sgn denotes the "sign" function ($= 1$ for $x \geq 0$ and -1 for $x < 0$). Then by Ito's formula

$$\begin{aligned} d|I_t|^p &= f'(I_t)dI_t + \frac{1}{2}f''(I_t)d\langle I \rangle_t \\ &= |I_s|^{p-1} \operatorname{sgn}(I_s)X_s dB_s + \frac{1}{2}p(p-1)|I_s|^{p-2}X_s^2 ds, \end{aligned}$$

i.e., as $I_0 = 0$,

$$|I_t|^p = p \int_0^t |I_s|^{p-1} \operatorname{sgn}(I_s)X_s dB_s + \frac{1}{2}p(p-1) \int_0^t |I_s|^{p-2}X_s^2 ds.$$

- 2) One can of course assume $X \in M^p([0, T])$, otherwise the statement is obvious (the right-hand side is $= +\infty$). Let $I_t = \int_0^t X_s dB_s$ and define $I_t^* = \sup_{0 \leq s \leq t} |I_s|$. $(I_t)_t$ is a square integrable martingale and by Doob's inequality

$$\mathbb{E} \left(\sup_{0 \leq t \leq T} \left| \int_0^t X_s dB_s \right|^p \right) = \mathbb{E}[I_T^{*p}] \leq \left(\frac{p}{p-1} \right)^p \mathbb{E}[|I_T|^p].$$

Let us now first assume $|I_T^*| \leq K$: this guarantees that $|I|^{p-1} \operatorname{sgn}(I)X \in M^2([0, T])$. Let us take the expectation recalling that the stochastic integral has zero mean. By Doob's inequality and Hölder's inequality with the exponents $\frac{p}{2}$ and $\frac{p}{p-2}$, we have

$$\begin{aligned} \mathbb{E}[I_T^{*p}] &\leq \left(\frac{p}{p-1} \right)^p \mathbb{E}[|I_T|^p] = \underbrace{\frac{1}{2} \left(\frac{p}{p-1} \right)^p p(p-1)}_{:=c_0} \mathbb{E} \left(\int_0^T |I_s|^{p-2} X_s^2 ds \right) \\ &\leq c_0 \mathbb{E} \left(I_T^{*p-2} \int_0^T X_s^2 ds \right) \leq c_0 \mathbb{E}[I_T^{*p}]^{1-\frac{2}{p}} \mathbb{E} \left[\left(\int_0^T X_s^2 ds \right)^{\frac{p}{2}} \right]^{\frac{2}{p}} \end{aligned}$$

As we assume $|I_T^*| \leq K$, $\mathbb{E}[I_T^{*p}] < +\infty$ and in the previous inequality we can divide by $\mathbb{E}[I_T^{*p}]^{1-\frac{2}{p}}$, which gives

$$\mathbb{E}[I_T^{*p}]^{\frac{2}{p}} \leq c_0 \mathbb{E} \left[\left(\int_0^T X_s^2 ds \right)^{\frac{p}{2}} \right]^{\frac{2}{p}}$$

i.e.

$$\mathbb{E}[I_T^{*p}] \leq c_0^{p/2} \mathbb{E} \left[\left(\int_0^T X_s^2 ds \right)^{\frac{p}{2}} \right] \leq c_0^{p/2} T^{\frac{p-2}{p}} \mathbb{E} \left[\left(\int_0^T |X_s|^p ds \right) \right].$$

- 3) Let $\tau_n = \inf\{t \leq T; |I_t| \geq n\}$ ($\tau(n) = T$ if $\{\} = \emptyset$). $(\tau_n)_n$ is a sequence of stopping times increasing to T , as the paths of I are continuous and then also bounded. We have therefore $I_{\tau_n \wedge t} \rightarrow I_t$ as $n \rightarrow \infty$ and

$$I_{t \wedge \tau_n} = \int_0^{t \wedge \tau_n} X_s dB_s = \int_0^t X_s 1_{\{s < \tau_n\}} dB_s,$$

so that one has

$$\mathbb{E}(I_{T \wedge \tau_n}^{*p}) \leq c_0^{p/2} \mathbb{E} \left[\left(\int_0^T |X_s|^2 1_{\{s < \tau_n\}} ds \right)^{\frac{p}{2}} \right] \leq c_0^{p/2} \mathbb{E} \left[\left(\int_0^T |X_s|^2 ds \right)^{\frac{p}{2}} \right]$$

and we can just apply Fatou's lemma. Finally, again by Hölder's inequality,

$$\mathbb{E} \left[\left(\int_0^T |X_s|^2 ds \right)^{\frac{p}{2}} \right] \leq T^{\frac{p-2}{p}} \mathbb{E} \left[\int_0^T |X_s|^p ds \right].$$

Exercise 6.

Consider the Itô process

$$X_t = X_0 + \int_0^t f_s ds + \int_0^t g_s dB_s = X_0 + J_t + I_t$$

where $\{B_t\}_t$ is a standard Brownian motion, $f \in M^1([0, T])$, and $g \in M^2([0, T])$.

Let us focus first on the process

$$I_t = \int_0^t g_s dB_s.$$

1) Show that I_t is a martingale.

2) Show that

$$\langle I \rangle_t = \int_0^t g_s^2 ds,$$

where $\langle I \rangle_t$ is the quadratic variation of I_t (i.e. that $I_t^2 - \langle I \rangle_t$ is a martingale).

It can be shown that

$$\langle I \rangle_t = \lim_{\pi \rightarrow 0} \sum_{j=0}^{n-1} (I_{t_{j+1}} - I_{t_j})^2 \text{ in probability,}$$

where $\pi = \{0 = t_0 < \dots < t_n = t\}$ is any partition of $[0, t]$ and $|\pi| = \max_j |t_{j+1} - t_j|$.

We focus now on the process X_t . By definition, $\langle X \rangle_t = \langle I \rangle_t = \int_0^t g_s^2 ds$.

3) Show that

$$\langle X \rangle_t = \lim_{\pi \rightarrow 0} \sum_{j=0}^{n-1} (X_{t_{j+1}} - X_{t_j})^2 \text{ in probability}$$

4) Consider another Itô process

$$Y_t = Y_0 + \int_0^t \tilde{f}_s ds + \int_0^t \tilde{g}_s dB_s = X_0 + \tilde{J}_t + \tilde{I}_t$$

with $\tilde{f} \in M^1([0, T])$, and $\tilde{g} \in M^2([0, T])$, by definition

$$\langle X, Y \rangle = \langle I, \tilde{I} \rangle_t = \int_0^t g_s \tilde{g}_s ds.$$

5) Show that

$$\langle X, Y \rangle = \lim_{\pi \rightarrow 0} \sum_{j=0}^{n-1} (X_{t_{j+1}} - X_{t_j})(Y_{t_{j+1}} - Y_{t_j}) \text{ in probability,}$$

using that

$$\langle I, \tilde{I} \rangle_t = \lim_{\pi \rightarrow 0} \sum_{j=0}^{n-1} (I_{t_{j+1}} - I_{t_j})(\tilde{I}_{t_{j+1}} - \tilde{I}_{t_j}) \text{ in probability.}$$

Solution

1) If $t > s$ one has

$$\mathbb{E}[I(t) - I(s) \mid \mathcal{F}_s] = \mathbb{E} \left[\underbrace{\int_s^t X_u dB_u}_{\mathcal{F}_s\text{-independent}} \mid \mathcal{F}_s \right] = 0 \quad \text{a.s.}$$

and therefore we have the martingale relation

$$\mathbb{E}[I(t) \mid \mathcal{F}_s] = I(s) + \mathbb{E}[I(t) - I(s) \mid \mathcal{F}_s] = I(s) \quad \text{a.s. .}$$

2) For the quadratic variation property we need to show that $I(t)^2 - \langle I \rangle_t$ is a martingale

$$\begin{aligned} \mathbb{E}[I(t)^2 - \langle I \rangle_t \mid \mathcal{F}_s] &= \mathbb{E}[(I(t) - I(s))^2 + 2((I(t) - I(s))I(s)) + I(t)^2 - (\langle I \rangle_t - \langle I \rangle_s) - \langle I \rangle_s \mid \mathcal{F}_s] \\ &= \mathbb{E}[(I(t) - I(s))^2 \mid \mathcal{F}_s] + 2\mathbb{E}[(I(t) - I(s))I(s) \mid \mathcal{F}_s] + I(s)^2 - \mathbb{E}[\int_s^t X_s^2 ds] - \langle I \rangle_s \\ &= \mathbb{E}[(\int_s^t X_s dB_s)^2] + I(s)^2 - \mathbb{E}[\int_s^t X_s^2 ds] - \langle I \rangle_s \\ &= I(s)^2 - \langle I \rangle_s. \end{aligned} \tag{6.1}$$

3) One has

$$\begin{aligned} \sum_{j=0}^{n-1} (X_{t_{j+1}} - X_{t_j})^2 &= \sum_{j=0}^{n-1} (\int_{t_j}^{t_{j+1}} f_s ds + \int_{t_j}^{t_{j+1}} g_s dB_s)^2 \\ &= \underbrace{\sum_{j=0}^{n-1} (\int_{t_j}^{t_{j+1}} f_s ds)^2}_{T_1} + 2 \underbrace{\sum_{j=0}^{n-1} (\int_{t_j}^{t_{j+1}} f_s ds)(\int_{t_j}^{t_{j+1}} g_s dB_s)}_{T_2} + \underbrace{\sum_{j=0}^{n-1} (\int_{t_j}^{t_{j+1}} g_s dB_s)^2}_{T_3} \end{aligned} \tag{6.2}$$

Since the function $t \rightarrow f_t(\cdot)$ is integrable in t ($f \in \mathcal{M}^1([0, T])$), let $N \in F$ be s.t. $\forall \omega \in N^C$ we have $N = \int_0^T f_s ds < \infty$ then $\forall \omega \in N^C$

$$\begin{aligned} |T_1(\omega)| &\leq \max_j |I_{t_{j+1}} - I_{t_j}| \sum_{j=0}^{n-1} |\int_{t_j}^{t_{j+1}} f_s ds| \\ &\leq \max_j |I_{t_{j+1}} - I_{t_j}| \int_0^t f_s ds \rightarrow 0 \end{aligned} \tag{6.3}$$

Hence $T_1 \rightarrow 0$ a.s. and in probability. Similarly for T_2

$$\begin{aligned} |T_2(\omega)| &\leq 2 \max_j |I_{t_{j+1}} - I_{t_j}| \int_{t_j}^{t_{j+1}} f_s ds \\ &\leq 2 \max_j |I_{t_{j+1}} - I_{t_j}| \int_0^t f_s ds \rightarrow 0 \end{aligned} \tag{6.4}$$

since I_t is continuous. Hence $T_2 \rightarrow 0$ a.s. and in probability. Finally, $T_3 \rightarrow \int_0^t g_s^2 ds$ in probability by L^2 convergence of the construction of the stochastic integral.

4) Analogous to Point 3).