Series 4 - October 9, 2024

Exercise 1.

Let $G \in \mathrm{M}^2(0,T)$ with $t \mapsto G(t,\omega)$ continuous for almost every ω and consider a partition $P_\Delta = \{0 = t_0 < t_1 < \ldots < t_{m_\Delta} = T\}$ of $[0,T]$ of size Δ, i.e., $t_j = j\Delta$ for $j = 0,\ldots,m_\Delta$. Assume that $\mathbb{E}[G(t)G(s)]$ is a continuous function of t and s. Show that

$$\lim_{\Delta\to 0} \sum_{j=1}^{m_\Delta} G(t_{j-1})(W(t_j) - W(t_{j-1})) = \int_0^T G(t)\,\mathrm{d}W(t) \quad \text{in } \mathrm{L}^2.$$

Exercise 2.

Let $(W(t), t \geqslant 0)$ be a one-dimensional Brownian motion. Without implement the Itô formula but using the construction of the stochastic integral, show that:

i) $\mathrm{d}(W^2) = 2W\mathrm{d}W + \mathrm{d}t$,

ii) $\mathrm{d}(tW) = W\mathrm{d}t + t\mathrm{d}W$.

Exercise 3.

Let $F\colon [a,b]\times\mathbb{R} \to \mathbb{R}$ be a continuous function with continuous partial derivatives $\partial F/\partial t, \partial F/\partial x, \partial^2 F/\partial x^2$ and let $(\partial F/\partial x)(t,x) = f(t,x)$. Show the following analogue of the fundamental theorem of the Leibniz–Newton calculus for the Itô calculus

$$\int_a^b f(t,W(t))\mathrm{d}W(t) = F(t,W(t))\Big|_a^b - \int_a^b \Big(\frac{\partial F}{\partial t}(t,W(t)) + \frac{1}{2}\frac{\partial f}{\partial x}(t,W(t))\Big)\mathrm{d}t.$$

In particular, when $F(t,x) = F(x)$ is independent of t, it reads

$$\int_a^b f(W(t))\mathrm{d}W(t) = F(W(t))\Big|_a^b - \frac{1}{2}\int_a^b f'(W(t))\mathrm{d}t.$$

Then, use this formula to compute

$$\int_0^t W(s)e^{W(s)}\mathrm{d}W(s).$$

Exercise 4.

Compute $\mathbb{E}\Big[B_s \int_0^t B_u \mathrm{d}B_u\Big]$ and $\mathbb{E}\Big[B_s^2\Big(\int_0^t B_u\mathrm{d}B_u\Big)^2\Big]$ for $0 \leqslant s \leqslant t$.
Recall: If $\xi \sim \mathcal{N}(0,1)$, then $\mathbb{E}[\xi^4] = 3$.

Exercise 5.

Let $X \in M^2([0,T])$ and consider the stochastic integral $I_t = \int_0^t X_s dB_s$.

1) Use Itô's formula to show that

$$|I_t|^p = p\int_0^t |I_s|^{|p-1} \operatorname{sgn}(I_s) X_s dB_s + \frac{1}{2}p(p-1)\int_0^t |I_s|^{p-2} X_s^2 ds.$$

2) Assume $|I_t| \leqslant K$ a.s. for some $K > 0$. Deduce that there exists $c_p = C(T,p)$ such that

$$\mathrm{E}\left(\sup_{0\leqslant t\leqslant T}\left|\int_0^t X_s dB_s\right|^p\right) \leqslant c_p \mathrm{E}\left[\left(\int_0^T |X_s|^2 ds\right)^{\frac{p}{2}}\right] \leqslant c_p T^{\frac{p-2}{2}} \mathrm{E}\left(\int_0^T |X_s|^p ds\right)$$

for some constant $c_p > 0$.

Hint. Use Doob's martingale and Holder inequalities.

3) Show that 2) works without asking for $|I_t| \leqslant K$ a.s. .

Hint. Take $\tau_n = \inf\{t \leqslant T; |I_t| \geqslant n\}$ and consider the martingale $I_{t\wedge\tau_n}$. Then take the limit for $n \to \infty$ and conclude by Fatou's lemma.

Exercise 6.

Consider the Itô process

$$X_t = X_0 + \int_0^t f_s\, ds + \int_0^t g_s\, dB_s = X_0 + J_t + I_t$$

where $\{B_t\}_t$ is a standard Brownian motion, $f \in M^1([0,T])$, and $g \in M^2([0,T])$.

Let us focus first on the process

$$I_t = \int_0^t g_s\, dB_s.$$

1) Show that I_t is a martingale.

2) Show that

$$\langle I\rangle_t = \int_0^t g_s^2\, ds,$$

where $\langle I\rangle_t$ is the quadratic variation of I_t (i.e. that $I_t^2 - \langle I\rangle_t$ is a martingale).

It can be shown that

$$\langle I\rangle_t = \lim_{\pi\to 0}\sum_{j=0}^{n-1}(I_{t_{j+1}} - I_{t_j})^2 \text{ in probability,}$$

where $\pi = \{0 = t_0 < \cdots < t_n = t\}$ is any partition of $[0,t]$ and $|\pi| = \max_j |t_{j+1} - t_j|$.

We focus now on the process X_t. By definition, $\langle X\rangle_t = \langle I\rangle_t = \int_0^t g_s^2\, ds$.

3) Show that

$$\langle X\rangle_t = \lim_{\pi\to 0}\sum_{j=0}^{n-1}(X_{t_{j+1}} - X_{t_j})^2 \text{ in probability}$$

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4) Consider another Itô process

$$Y_t = Y_0 + \int_0^t \tilde{f}_s\, ds + \int_0^t \tilde{g}_s\, dB_s = X_0 + \tilde{J}_t + \tilde{I}_t$$

with $\tilde{f} \in M^1([0,T])$, and $\tilde{g} \in M^2([0,T])$, by definition

$$\langle X, Y\rangle = \langle I, \tilde{I}\rangle_t = \int_0^t g_s \tilde{g}_s ds.$$

5) Show that

$$\langle X, Y\rangle = \lim_{\pi\to 0}\sum_{j=0}^{n-1}(X_{t_{j+1}} - X_{t_j})(Y_{t_{j+1}} - Y_{t_j}) \text{ in probability,}$$

using that

$$\langle I, \tilde{I}\rangle_t = \lim_{\pi\to 0}\sum_{j=0}^{n-1}(I_{t_{j+1}} - I_{t_j})(\tilde{I}_{t_{j+1}} - \tilde{I}_{t_j}) \text{ in probability.}$$