Series 3 - October 2, 2024

Exercise 1.

Consider a scalar standard Brownian motion (Wiener process) on $[0, 1]$.

i) Write a Matlab code to simulate a discretized Brownian motion $W(t)$ on $t_j = j\Delta t$ (by simulating the independent increments) with $\Delta t = 2^{-4}, 2^{-6}, 2^{-8}$, and compute the mean on all grid points over $20, 200, 2000$ trajectories. Verify that $\mathbb{E}(W(t)) = 0$.

ii) Compute the discretized stochastic process $X(t) = X_0 \exp((\lambda - \frac{1}{2}\mu^2)t + \mu W(t))$ on $t_j = j\Delta t$, for $\lambda = 2$, $\mu = 1$, $X_0 = 1$ with $\Delta t = 2^{-4}, 2^{-6}, 2^{-8}$, and compute the mean of $X(t)$ on all grid points over $20, 200, 2000$ trajectories. Can you guess what $\mathbb{E}(X(t))$ is?

Solution

1)

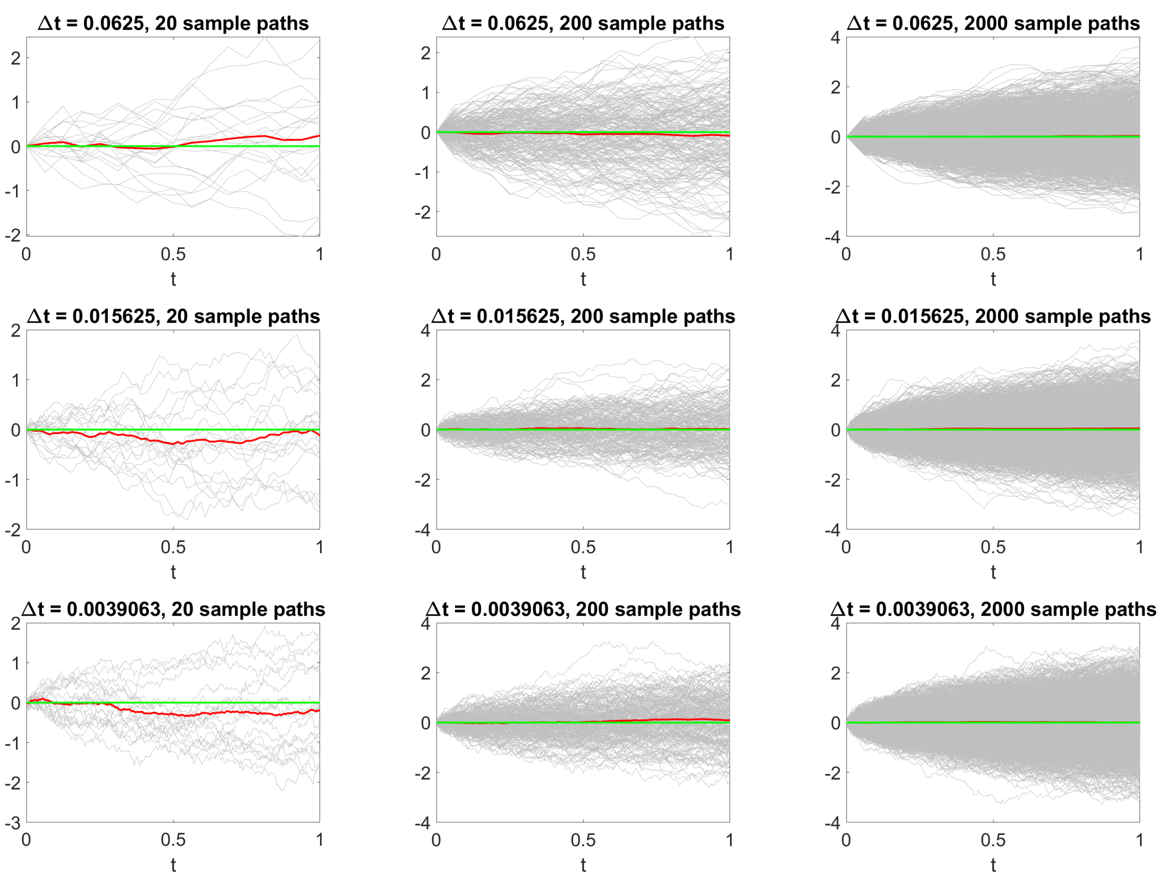

Figure 1: Discretized Brownian motion. The red and green lines are the empirical and exact averages, respectively.

2)

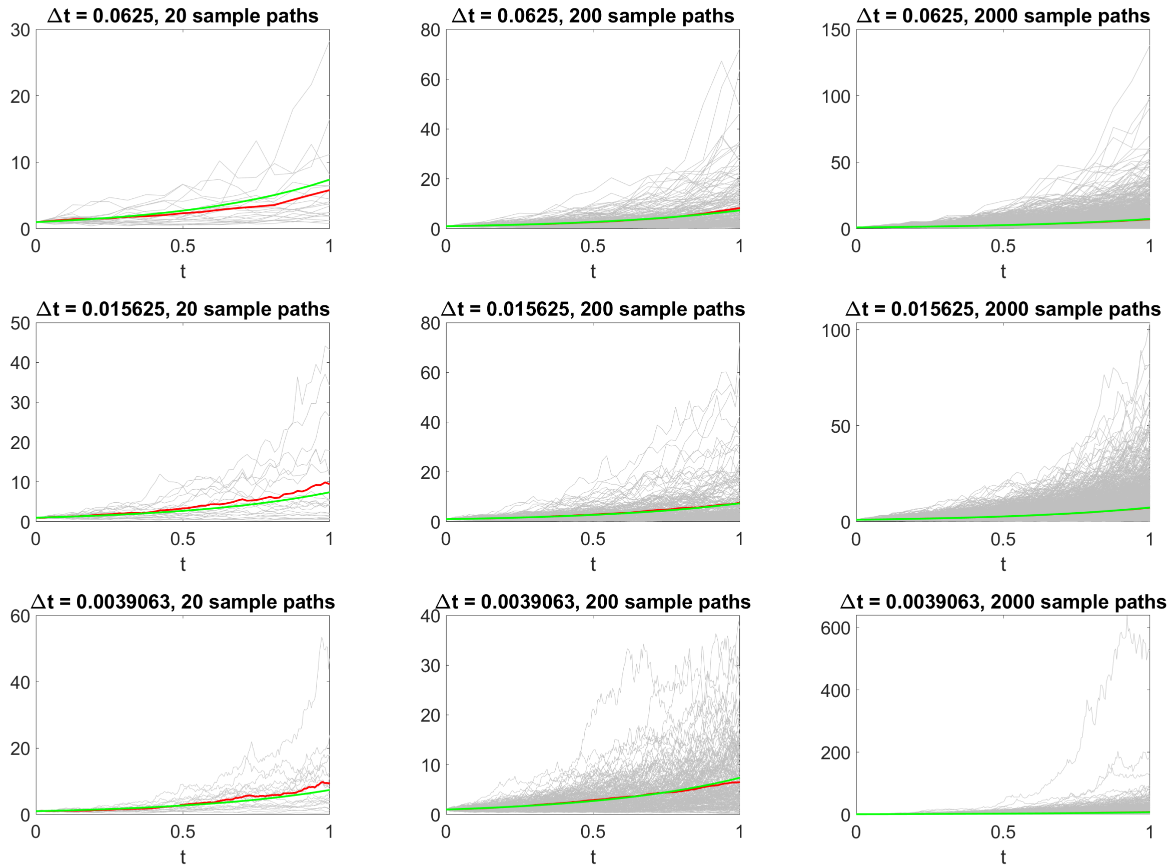

Figure 2: Discretized stochastic process. The red and green lines are the empirical and exact averages, respectively.

Exercise 2.

(Brownian bridge) Let $T > 0$ and consider the interval $[0, T]$. A Brownian bridge is a standard Gaussian process $(Z(t), 0 \leqslant t \leqslant T)$ such that

$$\mathrm{Cov}(Z(t), Z(s)) = \min\{s, t\} - \frac{st}{T}.$$

Let $(W(t), 0 \leqslant t \leqslant T)$ be a standard Brownian motion.

i) Show that $Z(t) = W(t) - \frac{t}{T} W(T)$ is a Brownian bridge.

In some applications it is useful to construct a modified Wiener process $(X(t), 0 \leqslant t \leqslant T)$ for which all sample paths satisfy $X(0) = x$ and $X(T) = y$ for some $x, y \in \mathbb{R}$.

ii) Using the Brownian bridge, construct such a Gaussian process with

$$\mathbb{E}[X(t)] = x - \frac{t}{T}(x - y) \qquad \text{and} \qquad \mathrm{Cov}(X(t), X(s)) = \min\{s, t\} - \frac{st}{T}. \tag{2.1}$$

iii) Simulate the stochastic process $(X(t), 0 \leqslant t \leqslant 2)$ constructed in point *ii)* with $X(0) = 1$ and $X(2) = 2$. Use different step sizes $\Delta t = 2^{-4}, 2^{-6}, 2^{-8}$ and approximate $\mathbb{E}[X(t)]$ over $M = 20, 200, 2000$ trajectories.

Solution

i) It is clear that Z is a standard Gaussian process. Furthermore, for any $0 \leqslant s, t \leqslant T$ we have

$$\begin{aligned}\mathbb{E}(Z(t)Z(s)) &= \mathbb{E}(W(t)W(s)) - \frac{t}{T}\mathbb{E}(W(T)W(s)) - \frac{s}{T}\mathbb{E}(W(T)W(t)) + \frac{st}{T^2}\mathbb{E}(W(T)^2) \\ &= \min\{s,t\} - \frac{st}{T}.\end{aligned}$$

ii) The desired Brownian bridge is given by $X(t) = x + W(t) - \frac{t}{T}(x + W(T) - y)$. As $W(t) \sim N(0,t)$, it is clear that $\mathbb{E}(Z(t)) = x - \frac{t}{T}(x-y)$. By definition of covariance and applying the properties of Brownian motion, we have

$$\begin{aligned}\mathrm{Cov}(Z(t), Z(s)) &= \mathbb{E}\Big((Z(t) - \mathbb{E}(Z(t)))(Z(s) - \mathbb{E}(Z(s)))\Big) \\ &= \mathbb{E}\Big((W(t) - \frac{t}{T}W(T))(W(s) - \frac{s}{T}W(T))\Big) = \min\{s,t\} - \frac{st}{T}.\end{aligned}$$

iii) The plots are given in Figure 3.

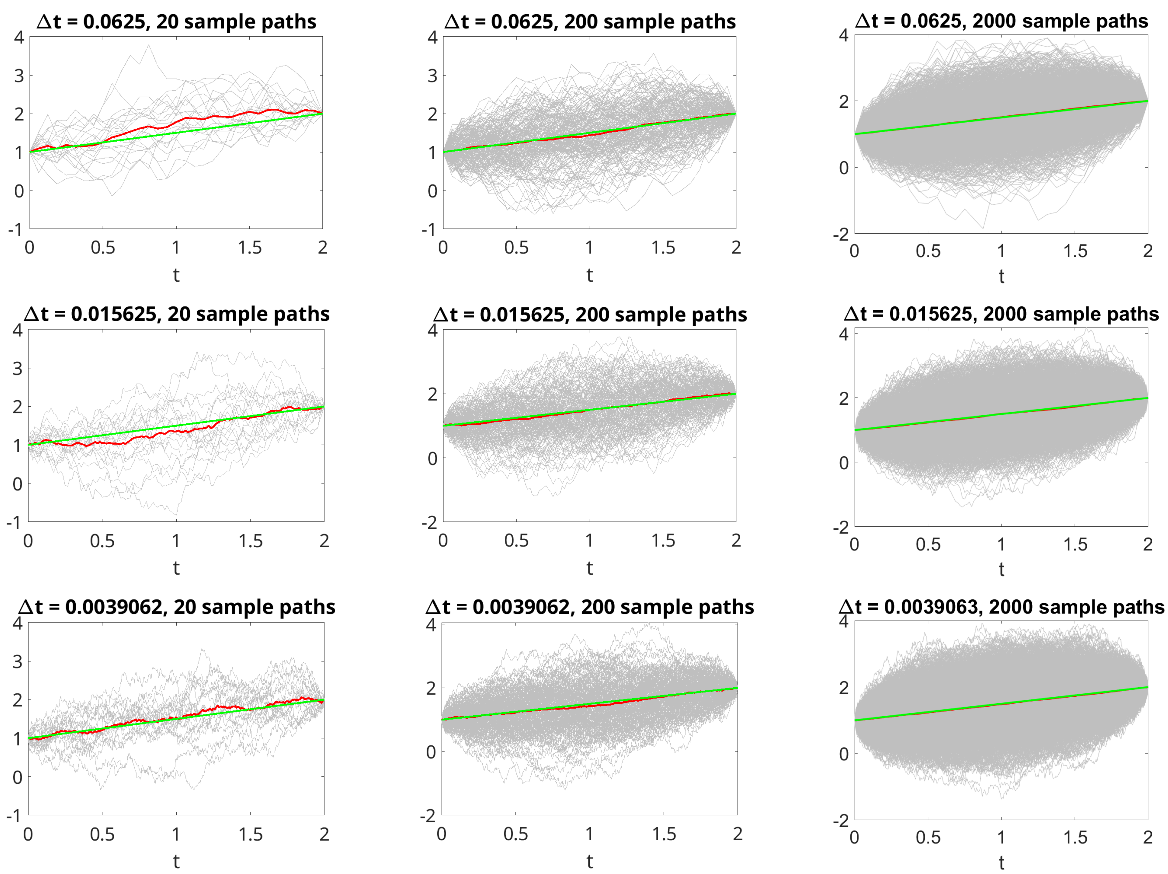

Figure 3: Brownian bridge. The red and green lines are the empirical and exact averages, respectively.

Exercise 3.

For a given $n \geqslant 1$, we approximate the Brownian motion as

$$W_n(t) = \sum_{k=0}^{2^{n+1}-1} s_k(t)\xi_k,$$

where $\{s_k\}_{k=0}^{2^{n+1}-1}$ are the Schauder functions defined in previous exercises and $\{\xi_k\}_{k=0}^{2^{n+1}-1}$ are independent standard Gaussian random variables $\xi_k \sim N(0,1)$. Furthermore, let $P = \{0 = t_0 < t_1 < \ldots < t_N = 1\}$ be the uniform partition of $[0,1]$ with $\Delta t = 2^{-12}$.

i) For $n = 3, 4, \ldots, 10$, plot W_n on the partition P and observe numerically that the sequence

$$V_n = \sum_{i=1}^{N} |W_n(t_i) - W_n(t_{i-1})|$$

diverges.

ii) Consider the series of the time derivative D_n of W_n

$$D_n(t) = W_n(t) = \sum_{k=0}^{2^{n+1}-1} h_k(t)\xi_k,$$

where $\{h_k\}_{k=0}^{2^{n+1}-1}$ are the Haar functions defined in previous exercises. For $n = 3, 4, \ldots, 10$, plot D_n on the partition P and observe numerically that the series diverges.

Solution

The plots are given in Figure 6.

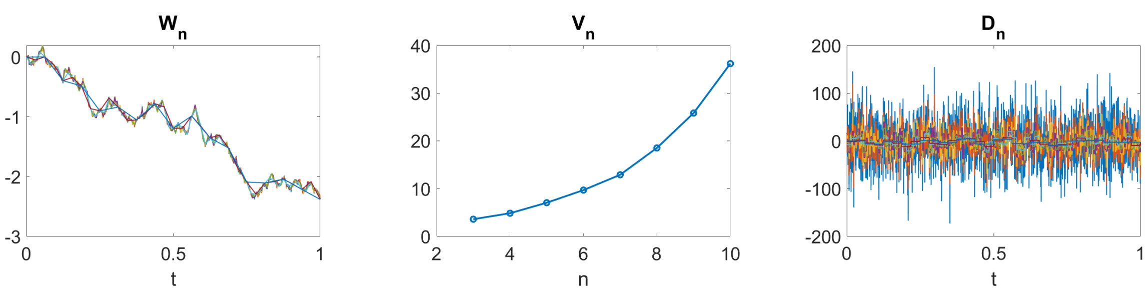

Figure 4: Quantities W_n, V_n and D_n for different values of $n = 3, 4, \ldots, 10$.

Exercise 4.

In some circumstances, we compute a discretized Brownian path $\{t_i, W_i\}_{i=0}^{L}$ with $\Delta = t_{i+1} - t_i$ and then wish to refine the discretization; that is, to compute values for the path at one or more times in between the set $\{t_i\}_{i=0}^{L}$. To be specific, suppose we need a new value $W_{i+\frac{1}{2}}$, to represent the path at time $t_{i+\frac{1}{2}} := \frac{1}{2}(t_i + t_{i+1})$. To be consistent, the new r.v. $W\Big(t_{i+\frac{1}{2}}\Big)$ has to satisfy all the properties of Brownian motion.

1) Show that

$$W\Big(t_{i+\frac{1}{2}}\Big) = \frac{1}{2}(W(t_i) + W(t_{i+1})) + Y_{i+\frac{1}{2}}, \quad \text{where } Y_{i+\frac{1}{2}} \sim N\Big(0, \frac{1}{4}\Delta t\Big) \tag{4.1}$$

with $Y_{i+\frac{1}{2}}$ independent of all other r.v. used to create the path, guarantees all the properties of a Brownian motion.

2) Generalize the formula (4.1) to the case where, given $W(t_i)$ and $W(t_{i+1})$, a value is needed for $W(t_i + \alpha\Delta t)$ for some $\alpha \in (0,1)$.

3) Simulate a Brownian motion W_t, where $t \in [0,4]$ for a mesh of $N = 101$ points, hence $h = 4 \cdot 10^{-2}$. Refine W with 201 points for $\alpha = \frac{1}{4}, \frac{1}{2}, \frac{3}{4}$ using the formula point 2) and for each value of α plot W and its refinement.

Solution

1) An obvious first try is to take the average of the neighboring values; so $W\left(t_{i+\frac{1}{2}}\right) = \frac{1}{2}(W(t_i) + W(t_{i+1}))$. This leads to

$$W(t_{i+1}) - W\left(t_{i+\frac{1}{2}}\right) = \frac{1}{2}(W(t_{i+1}) - W(t_i)) \sim \frac{1}{2}N(0, \Delta t) \sim N\left(0, \frac{1}{4}\Delta t\right)$$

and similarly, $W\left(t_{i+\frac{1}{2}}\right) - W(t_i) \sim N\left(0, \frac{1}{4}\Delta t\right)$, whereas in order to preserve second property on the refined mesh we require these increments to be $N\left(0, \frac{1}{2}\Delta t\right)$. The normal distribution is preserved under addition, and that variances add. This suggests taking

$$W\left(t_{i+\frac{1}{2}}\right) = \frac{1}{2}(W(t_i) + W(t_{i+1})) + Y_{i+\frac{1}{2}}, \qquad \text{where } Y_{i+\frac{1}{2}} \sim N\left(0, \frac{1}{4}\Delta t\right)$$

with $Y_{i+\frac{1}{2}}$ independent of all other random variables used to create the path. This gives

$$W(t_{i+1}) - W\left(t_{i+\frac{1}{2}}\right) \sim N\left(0, \frac{1}{2}\Delta t\right) \quad \text{and} \quad W\left(t_{i+\frac{1}{2}}\right) - W(t_i) \sim N\left(0, \frac{1}{2}\Delta t\right)$$

as required for the second properties. To respect the independence of Brownian increments, we must ensure that the new increments $W(t_{i+1}) - W\left(t_{i+\frac{1}{2}}\right)$ and $W\left(t_{i+\frac{1}{2}}\right) - W(t_i)$ are independent. Since both are normally distributed, this reduces to showing that the expected value of their product is the product of their expected values. Now, $\mathbb{E}\left[\left(W(t_{i+1}) - W\left(t_{i+\frac{1}{2}}\right)\right)\left(W\left(t_{i+\frac{1}{2}}\right) - W(t_i)\right)\right]$ has the form

$$\mathbb{E}\left[\left(\frac{W(t_{i+1}) - W(t_i)}{2} - Y_{i+\frac{1}{2}}\right)\left(\frac{W(t_{i+1}) - W(t_i)}{2} + Y_{i+\frac{1}{2}}\right)\right]$$

This simplifies to

$$\begin{aligned}\mathbb{E}\left[\left(\frac{W(t_{i+1}) - W(t_i)}{2}\right)^2 - Y^2_{i+\frac{1}{2}}\right] &= \mathbb{E}\left[\left(\frac{W(t_{i+1}) - W(t_i)}{2}\right)^2\right] - \mathbb{E}\left[Y^2_{i+\frac{1}{2}}\right] \\ &= \frac{\Delta t}{4} - \frac{\Delta t}{4} \\ &= 0\end{aligned}$$

as required. It follows that generates a $W\left(t_{i+\frac{1}{2}}\right)$ that preserves the three defining properties of Brownian motion.

Computationally, this implies that setting

$$W_{i+\frac{1}{2}} = \frac{1}{2}(W_i + W_{i+1}) + \frac{1}{2}\sqrt{\Delta t}\xi_i, \qquad \text{where } \xi_i \text{ is an independent } N(0,1) \text{ sample,}$$

allows us to ”fill in” a discretized Brownian path from resolution Δt to resolution $\frac{1}{2}\Delta t$.

2) We start by putting

$$W(t_i + \alpha\Delta t) = (1-\alpha)W(t_i) + \alpha W(t_{i+1})$$

and we compute $W(t_{i+1}) - W(t_i + \alpha\Delta t)$. We have

$$\begin{aligned} W(t_i + \alpha\Delta t) &= (1-\alpha)W(t_i) + \alpha W(t_{i+1}) + (1-\alpha)W(t_{i+1}) - (1-\alpha)W(t_{i+1}) \\ &= (1-\alpha)(W(t_i) - W(t_{i+1})) + W(t_{i+1}) \end{aligned}$$

Then

$$W(t_{i+1}) - W(t_i + \alpha\Delta t) = (1-\alpha)(W(t_{i+1}) - W(t_i)) \sim \mathcal{N}(0, (1-\alpha)^2\Delta t)$$

Since, by hypothesis of the Brownian motion $W(t_{i+1}) - W(t_i + \alpha\Delta t)$ has to be normally distributed with variance $(1-\alpha)\Delta t$, we have to find a normal random variable $Y_{i+a} \sim \mathcal{N}(0, k(\alpha)\Delta t)$ such that

$$(1-\alpha)^2 + k(\alpha) = 1 - \alpha$$

that is, $k(\alpha) = \alpha - \alpha^2$. Then, we put

$$W(t_i + \alpha\Delta t) = (1-\alpha)W(t_i) + \alpha W(t_{i+1}) + Y_{i+a}$$

where Y_{i+a} is an independent random variable normally distributed with zero mean and variance $\alpha - \alpha^2$. We have to verify the assumptions of the Brownian motion for $W(t_i + \alpha\Delta t)$ as in (10).

We have

$$W(t_i + 1) - W(t_i + \alpha\Delta t) = (1-\alpha)(W(t_{i+1}) - W(t_i)) - Y_{i+\alpha}$$

that is,

$$W(t_i + 1) - W(t_i + \alpha\Delta t) \sim \mathcal{N}(0, (1-\alpha)\Delta t)$$

and

$$W(t_i + \alpha\Delta t) - W(t_i) = \alpha(W(t_{i+1}) - W(t_i)) + Y_{i+a}$$

that is,

$$W(t_i + \alpha\Delta t) - W(t_i) \sim \mathcal{N}(0, \alpha\Delta t)$$

It remains to show that $W(t_{i+1}) - W(t_i + \alpha\Delta t)$ and $W(t_i + \alpha\Delta t) - W(t_i)$ are independent with $W(t_i + \alpha\Delta t)$ as in (10). We have that

$$\mathbb{E}[(W(t_{i+1}) - W(t_{i+\alpha})(W(t_i + \alpha) - W(t_i))]$$

assumes the form

$$\mathbb{E}[((1-\alpha)(W(t_{i+1}) - W(t_i)) - Y_{i+a})(\alpha(W(t_{i+1}) - W(t_i)) + Y_{i+a})]$$

Due to the independence of Y_{i+a}, the above expression simplifies to

$$\begin{aligned} \mathbb{E}\left[(1-\alpha)\alpha(W(t_{i+1}) - W(t_i))^2 - (Y_{i+\alpha})^2\right] &= (1-\alpha)\alpha\mathbb{E}\left[(W(t_{i+1}) - W(t_i))^2\right] \\ &\quad - \mathbb{E}\left[(Y_{i+\alpha})^2\right] \\ &= (1-\alpha)\alpha\Delta t - (\alpha - \alpha^2)\Delta t = 0 \end{aligned}$$

Therefore the two random variables are independent as required from the properties of Brownian motion.

Exercise 5.

3)

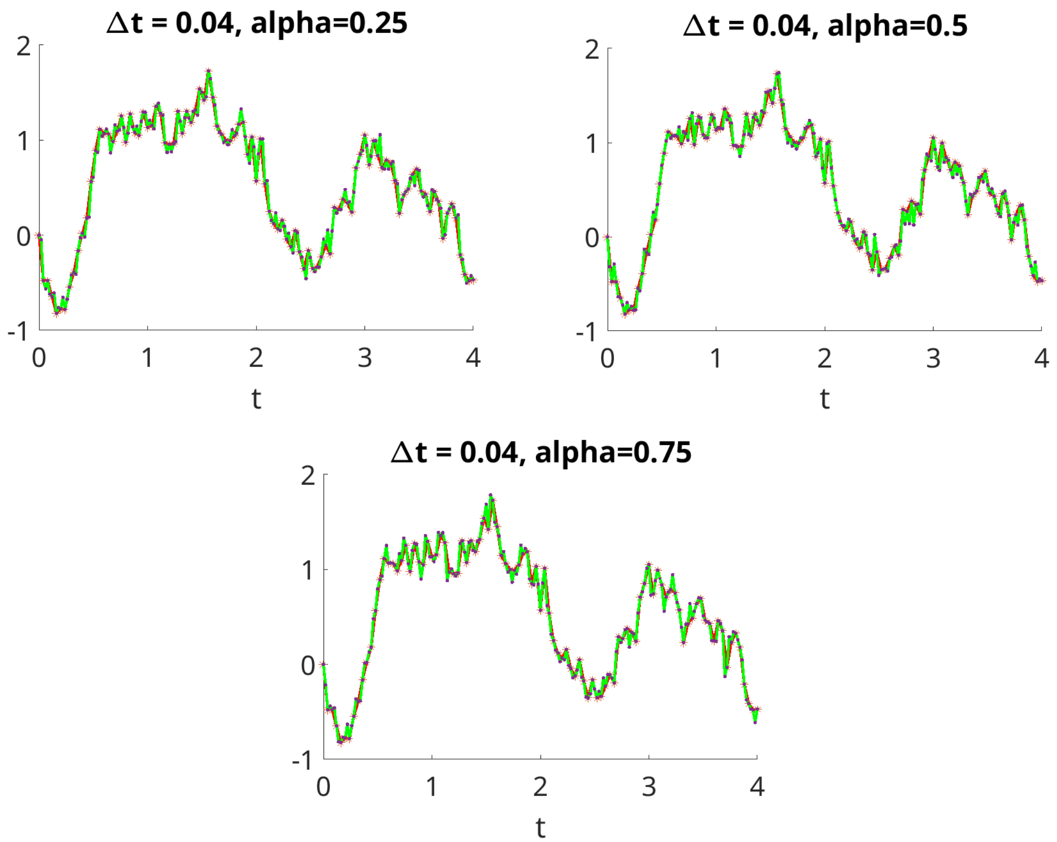

Figure 5: Brownian motion and its refinement for $\alpha = 0.25, 0.5, 0.75$.

Let $\{W(t)\}_{t\geqslant 0}$ be a Brownian motion with respect to a filtration $\{\mathcal{F}_t\}_{t\geqslant 0}$. Consider the quantity

$$I(P,\lambda) = \sum_{j=1}^{m} W(t_j^{\lambda})(W(t_j) - W(t_{j-1})), \tag{5.1}$$

where $P = \{0 = t_0 < t_1 < ... < t_m = t\}$ is a partition of $[0,t]$ of size Δ, i.e., $t_j = j\Delta$ for $j = 0, \dots, m$, and $t_j^{\lambda} = t_{j-1} + \lambda(t_j - t_{j-1})$ with $\lambda \in [0,1]$ for $j = 1, \dots, m$ are intermediate points. Define the quantity

$$I_{\lambda}(t) = \frac{1}{2}W(t)^2 + \left(\lambda - \frac{1}{2}\right)t.$$

1) Show that $I(P,\lambda) \to I_{\lambda}(t)$ in L^2 as $\Delta \to 0$.

2) Show that $I_{\lambda}(t)$ is a martingale with respect to the natural filtration if and only if $\lambda = 0$.

Solution

1) One can rewrite $I(P,\lambda)$ as

$$\begin{aligned} I(P,\lambda) =&\sum_{j=1}^{m} W(t_j^\lambda)(W(t_j)-W(t_{j-1})) \\ =&\frac{W(t)^2}{2} - \frac{1}{2}\underbrace{\sum_{j=1}^{m}(W(t_j)-W(t_{j-1}))^2}_{\mathrm{T}_1} + \underbrace{\sum_{j=1}^{m}(W(t_j)-W(t_j^\lambda))^2}_{\mathrm{T}_2} \\ &+ \underbrace{\sum_{j=1}^{m}(W(t_j)-W(t_j^\lambda))(W(t_j^\lambda)-W(t_{j-1}))}_{\mathrm{T}_3} \end{aligned} \tag{5.2}$$

We start proving that

$$\lim_{\Delta\to 0}\sum_{j=1}^{m}(W(t_j)-W(t_{j-1}))^2 = t, \text{ in } \mathrm{L}^2.$$

Indeed, one has

$$\left(\sum_{j=1}^{m}(W(t_j)-W(t_{j-1}))^2 - t\right) = \sum_{j=1}^{m}(W(t_j)-W(t_{j-1}))^2 - (t_j - t_{j-1}),$$

and thus, writing for the sake of notation $W(t_j)-W(t_{j-1}) = \Delta W_j$, it follows that

$$\begin{aligned} \mathbb{E}[(\sum_{j=1}^{m}(W(t_j)-W(t_{j-1}))^2 - t)] &= \sum_{k=1}^{m}\sum_{j=1}^{m}\mathbb{E}[((W(t_k)-W(t_{k-1}))^2-(t_k-t_{k-1}))((W(t_j)-W(t_{j-1}))^2-(t_j-t_{j-1}))] \\ &= \sum_{j=1,k=j}^{m}\mathbb{E}[((W(t_k)-W(t_{k-1}))^2-(t_k-t_{k-1}))((W(t_j)-W(t_{j-1}))^2-(t_j-t_{j-1}))] \\ &= \sum_{j=1}^{m}\mathbb{E}[(\Delta W_j)^4 - 2(\Delta W_j)^2(t_j-t_{j-1}) + (t_j-t_{j-1})^2] \\ &= \sum_{j=1}^{m}\mathbb{E}\left[\left(\left(\underbrace{\frac{\Delta W_j}{\sqrt{t_j-t_{j-1}}}}_{:=Y_j\sim\mathcal{N}(0,1)}\right)^2 - 1\right)^2\right](t_j-t_{j-1})^2 \\ &= \sum_{j=1}^{m}(\mathbb{E}[Y_j^4] - 2\mathbb{E}[Y_j^2] + 1)(t_j-t_{j-1})^2 \\ &= \sum_{j=1}^{m}(3-2+1)(t_j-t_{j-1})^2 \leqslant 2\Delta T \to 0, \text{ as } \Delta\to 0, \end{aligned} \tag{5.3}$$

using in the second line the independence of increments. Therefore, we have that $\mathrm{T}_1 \to t$ and $\mathrm{T}_2 \to \sum_{j=1}^{m}[\lambda(t_j-t_{j-1})+t_{j-1}] - t_{j-1} \to \lambda t$ in L^2 as $\Delta \to 0$. For T_3, defining $\Delta_\lambda W_j = (W(t_j) - W(t_j^\lambda))(W(t_j^\lambda)-W(t_{j-1}))$ and using the properties of Brownian motion we have that

$$\begin{aligned} \mathbb{E}[\mathrm{T}_3] &= \mathbb{E}[\sum_{j=1}^{m}(W(t_j)-W(t_j^\lambda))(W(t_j^\lambda)-W(t_{j-1})) + 2\sum_{j>k}\Delta_\lambda W_j \Delta_\lambda W_k] \\ &= \sum_{j=1}^{m}\mathbb{E}[(W(t_j)-W(t_j^\lambda))^2]\mathbb{E}[(W(t_j^\lambda)-W(t_{j-1}))^2] \\ &= \sum_{j=1}^{m}(t_j-t_j^\lambda)(t_j^\lambda-t_{j-1})^2 \\ &= \lambda(1-\lambda)\sum_{j=1}^{m}(t_j-t_{j-1})^2 \leqslant \lambda(1-\lambda)\sum_{j=1}^{m}(t_j-t_{j-1})\Delta \to 0, \text{ as } \Delta\to 0. \end{aligned} \tag{5.4}$$

Putting all together, we get the desired result.

2) First, by the triangle inequality we have

$$\mathbb{E}(|I_\lambda(t)|) \leqslant \tfrac{1}{2}\mathbb{E}(W(t)^2) + \big|\lambda - \tfrac{1}{2}\big|t \leqslant (1+\lambda)t < \infty,$$

which shows that $I_\lambda(t)$ is integrable for any $\lambda \in [0,1]$. By definition, $I_\lambda(t)$ is a martingale if and only if $\mathbb{E}(I_\lambda(t)|\mathcal{F}_s) = I_\lambda(s)$ a.s. for all $0 \leqslant s \leqslant t$. Let $0 \leqslant s \leqslant t$ and notice that

$$\mathbb{E}(I_\lambda(t)|\mathcal{F}_s) = \tfrac{1}{2}\mathbb{E}(W(t)^2|\mathcal{F}_s) + (\lambda - \tfrac{1}{2})t.$$

We rewrite the first term in the right-hand side as

$$\mathbb{E}(W(t)^2|\mathcal{F}_s) = \mathbb{E}((W(t)-W(s))^2\big|\mathcal{F}_s) + 2\mathbb{E}(W(s)(W(t)-W(s))\big|\mathcal{F}_s) + \mathbb{E}(W(s)^2\big|\mathcal{F}_s).$$

As $W(t) - W(s)$ is independent of $\mathcal{F}_s$ and $W(s)$ is $\mathcal{F}_s$-measurable, we have almost surely

$$\mathbb{E}(W(t)^2|\mathcal{F}_s) = \mathbb{E}((W(t)-W(s))^2) + 2W(s)\mathbb{E}(W(t)-W(s)) + W(s)^2 = t - s + W(s)^2.$$

Therefore, we obtain

$$\mathbb{E}(I_\lambda(t)|\mathcal{F}_s) = \tfrac{W(s)^2}{2} + \tfrac{1}{2}(t-s) + (\lambda - \tfrac{1}{2})t = \tfrac{W(s)^2}{2} + (\lambda - \tfrac{1}{2})s + \lambda(t-s) = I_\lambda(s) + \lambda(t-s),$$

and we conclude that $I_\lambda(t)$ is a martingale if and only if $\lambda = 0$.

Exercise 6.

Consider the Riemann sum

$$I(P,\lambda) = \sum_{j=1}^{m} W(t_j^\lambda)(W(t_j) - W(t_{j-1})), \tag{6.1}$$

and define the quantity

$$L(\lambda,t) = I(P,\lambda) - \frac{1}{2}W(t)^2.$$

From the theory (see previous exercise) we know that $L(\lambda,t)$ converges to $(\lambda - \frac{1}{2})t$ in $\mathrm{L}^2(\Omega)$. For $t = 1,2,3$ and $\lambda = 0, 1/4, 1/2, 3/4, 1$ approximate $\lim_{\Delta\to 0} L(\lambda,t)$ and verify the theoretical result. Choose $\Delta = 2^{-8}$ and use $M = 1000$ sample paths.

Solution

The plot is given in Figure 6.

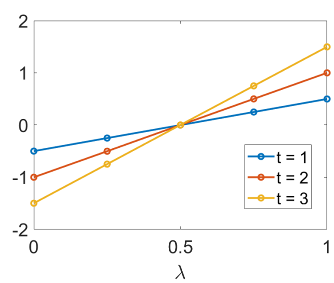

Figure 6: Approximation of $\lim_{\Delta\to 0} L(\lambda, t)$ in Exercise 5 for different values of λ and t.