Series 3 - October 2, 2024

Exercise 1.

Consider a scalar standard Brownian motion (Wiener process) on $[0, 1]$.

i) Write a Matlab code to simulate a discretized Brownian motion $W(t)$ on $t_j = j\Delta t$ (by simulating the independent increments) with $\Delta t = 2^{-4}, 2^{-6}, 2^{-8}$, and compute the mean on all grid points over $20, 200, 2000$ trajectories. Verify that $\mathbb{E}(W(t)) = 0$.

ii) Compute the discretized stochastic process $X(t) = X_0 \exp((\lambda - \frac{1}{2}\mu^2)t + \mu W(t))$ on $t_j = j\Delta t$, for $\lambda = 2$, $\mu = 1$, $X_0 = 1$ with $\Delta t = 2^{-4}, 2^{-6}, 2^{-8}$, and compute the mean of $X(t)$ on all grid points over $20, 200, 2000$ trajectories. Can you guess what $\mathbb{E}(X(t))$ is?

Exercise 2.

(Brownian bridge) Let $T > 0$ and consider the interval $[0, T]$. A Brownian bridge is a standard Gaussian process $(Z(t), 0 \leqslant t \leqslant T)$ such that

$$\mathrm{Cov}(Z(t), Z(s)) = \min\{s, t\} - \frac{st}{T}.$$

Let $(W(t), 0 \leqslant t \leqslant T)$ be a standard Brownian motion.

i) Show that $Z(t) = W(t) - \frac{t}{T}W(T)$ is a Brownian bridge.

In some applications it is useful to construct a modified Wiener process $(X(t), 0 \leqslant t \leqslant T)$ for which all sample paths satisfy $X(0) = x$ and $X(T) = y$ for some $x, y \in \mathbb{R}$.

ii) Using the Brownian bridge, construct such a Gaussian process with

$$\mathbb{E}[X(t)] = x - \frac{t}{T}(x - y) \qquad \text{and} \qquad \mathrm{Cov}(X(t), X(s)) = \min\{s, t\} - \frac{st}{T}. \tag{2.1}$$

iii) Simulate the stochastic process $(X(t), 0 \leqslant t \leqslant 2)$ constructed in point *ii)* with $X(0) = 1$ and $X(2) = 2$. Use different step sizes $\Delta t = 2^{-4}, 2^{-6}, 2^{-8}$ and approximate $\mathbb{E}[X(t)]$ over $M = 20, 200, 2000$ trajectories.

Exercise 3.

For a given $n \geqslant 1$, we approximate the Brownian motion as

$$W_n(t) = \sum_{k=0}^{2^{n+1}-1} s_k(t)\xi_k,$$

where $\{s_k\}_{k=0}^{2^{n+1}-1}$ are the Schauder functions defined in previous exercises and $\{\xi_k\}_{k=0}^{2^{n+1}-1}$ are independent standard Gaussian random variables $\xi_k \sim N(0, 1)$. Furthermore, let $P = \{0 = t_0 < t_1 < \ldots < t_N = 1\}$ be the uniform partition of $[0, 1]$ with $\Delta t = 2^{-12}$.

i) For $n = 3, 4, \dots, 10$, plot W_n on the partition P and observe numerically that the sequence

$$V_n = \sum_{i=1}^{N} |W_n(t_i) - W_n(t_{i-1})|$$

diverges.

ii) Consider the series of the time derivative D_n of W_n

$$D_n(t) = \dot{W}_n(t) = \sum_{k=0}^{2^{n+1}-1} h_k(t)\xi_k,$$

where $\{h_k\}_{k=0}^{2^{n+1}-1}$ are the Haar functions defined in previous exercises. For $n = 3, 4, \dots, 10$, plot D_n on the partition P and observe numerically that the series diverges.

Exercise 4.

In some circumstances, we compute a discretized Brownian path $\{t_i, W_i\}_{i=0}^{L}$ with $\Delta = t_{i+1} - t_i$ and then wish to refine the discretization; that is, to compute values for the path at one or more times in between the set $\{t_i\}_{i=0}^{L}$. To be specific, suppose we need a new value $W_{i+\frac{1}{2}}$, to represent the path at time $t_{i+\frac{1}{2}} := \frac{1}{2}(t_i + t_{i+1})$. To be consistent, the new r.v. $W\left(t_{i+\frac{1}{2}}\right)$ has to satisfy all the properties of Brownian motion.

1) Show that
$$W\left(t_{i+\frac{1}{2}}\right) = \frac{1}{2}(W(t_i) + W(t_{i+1})) + Y_{i+\frac{1}{2}}, \qquad \text{where } Y_{i+\frac{1}{2}} \sim N\left(0, \frac{1}{4}\Delta t\right) \tag{4.1}$$
with $Y_{i+\frac{1}{2}}$ independent of all other r.v. used to create the path, guarantees all the properties of a Brownian motion.

2) Generalize the formula (4.1) to the case where, given $W(t_i)$ and $W(t_{i+1})$, a value is needed for $W(t_i + \alpha\Delta t)$ for some $\alpha \in (0, 1)$.

3) Simulate a Brownian motion W_t, where $t \in [0, 4]$ for a mesh of $N = 101$ points, hence $h = 4 \cdot 10^{-2}$. Refine W with 201 points for $\alpha = \frac{1}{4}, \frac{1}{2}, \frac{3}{4}$ using the formula point 2) and for each value of α plot W and its refinement.

Exercise 5.

Let $\{W(t)\}_{t \geqslant 0}$ be a Brownian motion with respect to a filtration $\{\mathcal{F}_t\}_{t \geqslant 0}$. Consider the quantity

$$I(P, \lambda) = \sum_{j=1}^{m} W(t_j^{\lambda})(W(t_j) - W(t_{j-1})), \tag{5.1}$$

where $P = \{0 = t_0 < t_1 < \dots < t_m = t\}$ is a partition of $[0, t]$ of size Δ, i.e., $t_j = j\Delta$ for $j = 0, \dots, m$, and $t_j^{\lambda} = t_{j-1} + \lambda(t_j - t_{j-1})$ with $\lambda \in [0, 1]$ for $j = 1, \dots, m$ are intermediate points. Define the quantity

$$I_{\lambda}(t) = \frac{1}{2}W(t)^2 + \left(\lambda - \frac{1}{2}\right)t.$$

1) Show that $I(P, \lambda) \to I_{\lambda}(t)$ in L^2 as $\Delta \to 0$.

2) Show that $I_{\lambda}(t)$ is a martingale with respect to the natural filtration if and only if $\lambda = 0$.

Exercise 6.

Consider the Riemann sum

$$I(P,\lambda) = \sum_{j=1}^{m} W(t_j^{\lambda})(W(t_j) - W(t_{j-1})), \tag{6.1}$$

and define the quantity

$$L(\lambda, t) = I(P,\lambda) - \frac{1}{2}W(t)^2.$$

From the theory (see previous exercise) we know that $L(\lambda, t)$ converges to $(\lambda - \frac{1}{2})t$ in $\mathrm{L}^2(\Omega)$. For $t = 1, 2, 3$ and $\lambda = 0, 1/4, 1/2, 3/4, 1$ approximate $\lim_{\Delta \to 0} L(\lambda, t)$ and verify the theoretical result. Choose $\Delta = 2^{-8}$ and use $M = 1000$ sample paths.