Series 2 - September 25, 2024

Exercise 1.

Let $\left(\Omega, \mathcal{F}, P, (\mathcal{F}_t)_{t\geqslant 0}\right)$ be a stochastic basis. Consider two stochastic processes $X, Y : \Omega \to \mathbb{R}^T$, with T an interval of $\mathbb{R}^+$ or the whole $\mathbb{R}^+$, both adapted to $(\mathcal{F}_t)_{t\geqslant 0}$. Show that if X, Y are a modification of each other and they are a.s. continuous, then they are indistinguishable

Solution

As the paths of the two processes are a.s. continuous except eventually for a negligible event, if they coincide at the times of a dense subset $D \subset T$, they necessarily coincide on the whole T. Let $D = \{t_1, t_2, ...\}$ be a sequence of times which is dense in T(e.g. $T \cap \mathbb{Q}$). Then

$$\{X_t = Y_t \text{ for every } t\} = \bigcap_{t\in T}\{X_t = Y_t\} = \bigcap_{t_i \in D}\{X_{t_i} = Y_{t_i}\}$$

As $\mathrm{P}(X_{t_i} = Y_{t_i}) = 1$ for every i, it follows that

$$\begin{aligned} &\mathrm{P}(X_{t_i} \neq Y_{t_i}) \neq 0 \\ \Longrightarrow\ &\mathrm{P}(\cup\{X_{t_i} \neq Y_{t_i}\}) = 0 \quad \text{(countable union of zero probability events)} \\ \Longrightarrow\ &\mathrm{P}\left(\bigcap_{t_i\in D}\{X_{t_i} = Y_{t_i}\}\right) = 1 - \mathrm{P}(\cup\{X_{t_i} \neq Y_{t_i}\}) = 1. \end{aligned} \tag{1.1}$$

so that also $\mathrm{P}(X_t = Y_t \text{ for every } t) = 1$ and the two processes are indistinguishable.

Exercise 2.

Let $(X(t), 0 \leqslant t \leqslant T)$ and $(Y(t), 0 \leqslant t \leqslant T)$ be two stochastic processes. By providing counterexamples, show that:

i) $P(X(t) = Y(t)\ \forall\, t \in \mathbb{Q} \cap [0, T]) = 1 \quad \not\Longrightarrow \quad P(X(t) = Y(t)\ \forall\, t \in [0, T]) = 1.$

ii) $P(X(t) = Y(t)) = 1\ \forall\, t \in [0, T] \quad \not\Longrightarrow \quad P(X(t) = Y(t)\ \forall\, t \in [0, T]) = 1.$

(In view of the previous exercise, X, Y cannot be both a.s. continuous).

Solution

i) Consider $X(t)$ a stochastic process defined on $[0, T]$ and define

$$Y(t) = \begin{cases} X(t) & t \in [0, T] \cap \mathbb{Q}, \\ X(t) + 1 & t \in [0, T] \setminus \mathbb{Q}. \end{cases}$$

Then we have $P(X(t) = Y(t)\ \forall t \in \mathbb{Q} \cap [0, T]) = 1$ but $P(X(t) = Y(t)\ \forall t \in [0, T]) = 0$.

ii) Consider a uniform random variable $U \sim \mathcal{U}(0, T)$ on $[0, T]$. Then, define $X(t, \omega) = 0$ for all $t \in [0, T]$ and $\omega \in \Omega$ and $Y(t, \omega)$ as

$$Y(t, \omega) = \begin{cases} 1 & \text{if } t = U(\omega), \\ 0 & \text{otherwise.} \end{cases}$$

Then we have for all $t \in [0,T]$

$$P(X(t) = Y(t)) = P(Y(t) = 0) = P(U \neq t) = 1,$$

hence the left hand side of the implication is verified. Nonetheless, for all $\omega \in \Omega$ the paths $t \mapsto X(t,\omega)$ and $t \mapsto Y(t,\omega)$ are not equal as

$$Y(U(\omega),\omega) = 1 \neq 0 = X(U(\omega),\omega),$$

hence $P(X(t) = Y(t)\ \forall t \in [0,T]) = 0$.

Exercise 3.

Let B be a real Brownian motion and $\pi = \{t_0, \dots, t_m\}$ with $0 \leqslant s = t_0 < t_1 \cdots < t_m = t$ be a partition of the interval $[s,t]$, with $|\pi| = \max_{0\leqslant k\leqslant m-1}|t_{k+1} - t_k|$.

1) Show that

$$V_\pi^2 = \sum_{k=0}^{m-1} \left|B_{t_{k+1}} - B_{t_k}\right|^2$$

satisfies

$$\lim_{|\pi|\to 0^+} V_\pi^2 = t - s \quad \text{in } L^2.$$

2) Show that $\lim\limits_{|\pi|\to 0^+} V_\pi^1 = \infty$ a.s. $\forall t > s$, i.e. the paths of a Brownian motion do not have finite variation in any time interval a.s.

Solution

1) We have $\sum_{k=0}^{m-1}(t_{k+1} - t_k) = (t_1 - s) + (t_2 - t_1) + \cdots + (t - t_{m-1}) = t - s$ so we can write

$$V_\pi^2 - (t-s) = \sum_{k=0}^{m-1} \Big[\big(B_{t_{k+1}} - B_{t_k}\big)^2 - (t_{k+1} - t_k)\Big].$$

We must prove that $\mathrm{E}\Big[\big(V_\pi^2 - (t-s)\big)^2\Big] \to 0$ as $|\pi| \to 0$. Note that the random variable $v_k = \big(B_{t_{k+1}} - B_{t_k}\big)^2 - (t_{k+1} - t_k)$, $k = 0, \dots, m-1$ are independent (the increments of a Brownian motion over disjoint intervals are independent) and centered; therefore, if $h \neq k$ we have

$$\mathbb{E}\Big(\Big[\big(B_{t_h+1} - B_{t_h}\big)^2 - (t_{h+1} - t_h)\Big]\Big[\big(B_{t_{k+1}} - B_{t_k}\big)^2 - (t_{k+1} - t_k)\Big]\Big) = 0,$$

so that

$$\begin{aligned}
\mathrm{E}\Big[\big(V_\pi^2 - (t-s)\big)^2\Big] &= \mathrm{E}\left(\sum_{k=0}^{m-1}\Big[\big(B_{t_{k+1}} - B_{t_k}\big)^2 - (t_{k+1} - t_k)\Big] \times \sum_{h=0}^{m-1}\Big[\big(B_{t_{h+1}} - B_{t_h}\big)^2 - (t_{h+1} - t_h)\Big]\right) \\
&= \sum_{k=0}^{m-1} \mathrm{E}\Big(\Big[\big(B_{t_{k+1}} - B_{t_k}\big)^2 - (t_{k+1} - t_k)\Big]^2\Big) = \sum_{k=0}^{m-1}(t_{k+1} - t_k)^2 \mathrm{E}\left[\left(\frac{\big(B_{t_{k+1}} - B_{t_k}\big)^2}{t_{k+1} - t_k} - 1\right)^2\right].
\end{aligned} \tag{3.1}$$

But for every k the r.v. $\frac{B_{t_{k+1}} - B_{t_k}}{\sqrt{t_{k+1} - t_k}}$ is $N(0,1)$-distributed and the quantities

$$c = \mathrm{E}\left[\left(\frac{\big(B_{t_{k+1}} - B_{t_k}\big)^2}{t_{k+1} - t_k} - 1\right)^2\right]$$

are finite and do not depend on k ($c = 2$, if you really want to compute it...). Therefore, as $|\pi| \to 0^+$,

$$\mathrm{E}\Big[\big(V_\pi^2 - (t-s)\big)^2\Big] = c\sum_{k=0}^{m-1}(t_{k+1}-t_k)^2 \leqslant c|\pi|\sum_{k=0}^{m-1}|t_{k+1}-t_k| = c|\pi|(t-s) \to 0,$$

which yields the thesis.

2) Observe that

$$V_\pi^2 = \sum_{k=0}^{m-1}\big|B_{t_{k+1}} - B_{t_k}\big|^2 \leqslant \max_{0\leqslant i\leqslant m-1}\big|B_{t_{i+1}} - B_{t_i}\big| \sum_{k=0}^{m-1}\big|B_{t_{k+1}} - B_{t_k}\big|$$

As the paths are a.s. continuous, $\max_{0\leqslant i\leqslant m-1}\big|B_{t_{i+1}} - B_{t_i}\big| \to 0$ as $|\pi| \to 0^+$ and therefore if the paths had finite variation on $[s,t]$ for ω in some event A of positive probability, then on A we would have

$$\lim_{|\pi|\to 0^+}\sum_{k=0}^{m-1}\big|B_{t_{k+1}} - B_{t_k}\big| < +\infty$$

and therefore, taking the limit, we would have $\lim_{|\pi|\to 0^+} V_\pi^2(\omega) = 0$ on A, in contradiction with point 1).

Exercise 4.

Consider a Brownian motion B on a filtered probability space $\big(\Omega, \mathcal{F}, P, (\mathcal{F}_t)_{t\geqslant 0}\big)$. Show that

1) for every $0 \leqslant s < t$ the r.v. $B_t - B_s$ is independent of B_u, $\forall u \leqslant s$;

2) B is a Gaussian process.

Hint. To show that a stochastic processes $\{X_t\}_{t\in[0,\infty)}$ is a Gaussian it is enough to show that for any $t_1, t_2, \dots, t_m \in \mathbb{R}^+$ and any $\alpha_1, \alpha_2, \dots, \alpha_m \in \mathbb{R}$ the random variable $Z = \sum_{i=1}^m \alpha_i X_{t_i}$ is Gaussian.

Solution

1) From the definition of Brownian motion we know that B_u is $\mathcal{F}_u$ measurable, hence $\mathcal{F}_s$ measurable since $F_u \subset F_s$. This implies that $\sigma(B_u) \subset \mathcal{F}_s$. Again, by definition of Brownian motion $B_t - B_s$ is independent of $\mathcal{F}_s$, hence it is also independent of $\sigma(B_u)$, for all $u \leqslant s$, which is equivalent to say that $B_t - B_s$ id independent of B_u for all $u \leqslant s$.

2) One has to prove that the joint distributions of $B_{t_1}, \dots, B_{t_m}$ is Gaussian. The proof follows by induction. This is obvious if $m = 1$, with $s = 0$ one has $B_t \sim N(0,t)$ via definition of Brownian motion.

   Let us assume that $\sum_{i=1}^m \alpha_i B_{s_i}$, for any $\alpha_1, \dots, \alpha_m \in \mathbb{R}$ and any $s_1, s_2, \cdots, s_m \in \mathbb{R}^+$, is Gaussian and consider $Y = \sum_{i=1}^{m+1} \alpha_i B_{t_i}$ with w.l.o.g. $0 \leqslant t_1 < t_2 < \cdots < t_{m+1}$. Then, we can write

$$Y = \alpha_1 B_{t_1} + \cdots + \alpha_{m+1} B_{t_{m+1}} = \underbrace{\big[\alpha_1 B_{t_1} + \cdots + (\alpha_m + \alpha_{m+1}) B_{t_m}\big]}_{\tilde{Y}} + \alpha_{m+1}\big(B_{t_{m+1}} - B_{t_m}\big)$$

   By induction assumption $\tilde{Y}$ is Gaussian and $B_{t_{m+1}} - B_{t_m}$ is Gaussian by definition and independent of $\tilde{Y}$ via point 1). This yields the thesis.

Exercise 5.

The family of Haar functions $\{h_k\}_{k\geqslant 0}$ is defined for $0\leqslant t\leqslant 1$ as

$$h_0(t)=1, \qquad h_1(t)=\begin{cases} 1 & \text{if } 0\leqslant t\leqslant 1/2, \\ -1 & \text{if } 1/2<t\leqslant 1, \end{cases}$$

and for $2^n\leqslant k<2^{n+1}$ with $n=1,2,...$ as

$$h_k(t)=\begin{cases} 2^{n/2} & \text{if } \frac{k-2^n}{2^n}\leqslant t\leqslant \frac{k-2^n+1/2}{2^n}, \\ -2^{n/2} & \text{if } \frac{k-2^n+1/2}{2^n}<t\leqslant \frac{k-2^n+1}{2^n}, \\ 0 & \text{otherwise.} \end{cases} \qquad n=\lfloor \log_2 k\rfloor$$

i) Show that $\{h_k\}_{k\geqslant 0}$ is orthonormal in $\mathrm{L}^2(0,1)$.

ii) Show that $\{h_k\}_{k\geqslant 0}$ is complete in $\mathrm{L}^2(0,1)$, i.e., $f=\sum_{k=0}^{\infty}\langle f,h_k\rangle h_k$ in $\mathrm{L}^2(0,1)$ for any $f\in\mathrm{L}^2(0,1)$.
Hint. First prove that if $\langle g,h_k\rangle=0$ for all $k\geqslant 0$ then $g=0$ a.s. by showing that $\int_s^t g=0$ for all $0\leqslant s\leqslant t\leqslant 1$.

Solution

(See Evans chapter 3 for more details.)

i) First, for any $k\geqslant 0$ it holds $\langle h_k,h_k\rangle=2^{-n}(2^{n/2})^2=1$. Then, note that if $2^n\leqslant k<l<2^{n+1}$ for a $n\geqslant 1$, the support of h_k and h_l are disjoint and hence $\langle h_k,h_l\rangle=0$. Finally, if $2^{n_1}\leqslant k<2^{n_1+1}\leqslant 2^{n_2}\leqslant l<2^{n_2+1}$ for $1\leqslant n_1<n_2$, then the support of h_l is included in the half-support of h_k, hence $h_kh_l=\pm 2^{n_2/2}h_k$ and hence $\langle h_k,h_l\rangle=0$.

ii) Assume that $\langle f,h_k\rangle=0$ for all $k\geqslant 0$. First, it holds $\langle f,h_0\rangle=\int_0^1 f=0$. Then, we have $\langle f,h_1\rangle=\int_0^{1/2}f-\int_{1/2}^1 f=0$ and that implies $\int_0^{1/2}f=\int_{1/2}^1 f$ which due to the previous step gives $2\int_0^{1/2}f=\int_0^1 f=0$ and we conclude that $\int_0^{1/2}f=\int_{1/2}^1 f=0$. Continuing with the same reasoning we can show that for any $n\geqslant 1$ and for all $0\leqslant k<2^n+1$ we have $\int_{k/2^{n+1}}^{(k+1)/2^{n+1}} f=0$. Hence, by density of the extrema in $[0,1]$ we deduce that for any $0\leqslant t,s\leqslant 1$ it holds $\int_s^t f=0$, which implies that $f=0$ a.e. in $[0,1]$. Now let $f\in\mathrm{L}^2(0,1)$. Note that $g=f-\sum_{k=0}^{\infty}\langle f,h_k\rangle h_k$ satisfies $\langle g,h_j\rangle=0$ for all $j\geqslant 0$, and therefore $g=0$ a.e. and $f=\sum_{k=0}^{\infty}\langle f,h_k\rangle h_k$ in $\mathrm{L}^2(0,1)$.

Exercise 6.

The family of Schauder functions $\{s_k\}_{k\geqslant 0}$ is defined for $0\leqslant t\leqslant 1$ as $s_k(t)=\langle\chi_{[0,t]},h_k\rangle$, where $\{h_k\}_{k\geqslant 0}$ are the Haar functions and $\langle\cdot,\cdot\rangle$ is the inner product in $\mathrm{L}^2(0,1)$. Then let $W(t)=\sum_{k=0}^{\infty}\xi_k s_k(t)$, where $\{\xi_k\}_{k\geqslant 0}$ is a sequence of independent standard Gaussian random variables $\xi_k\sim N(0,1)$.

i) Show that $\sum_{k=0}^{\infty}s_k(r)s_k(t)=\min\{r,t\}$ for all $0\leqslant r,t\leqslant 1$.

ii) Show that there exists a constant $C>0$ such that for all $k\geqslant 2$ it holds

$$P(|\xi_k|>4\sqrt{\log k})\leqslant Ck^{-4},$$

and deduce that almost surely there exists a positive integer $\bar{k}$ such that for all $k>\bar{k}$ it holds $|\xi_k|\leqslant 4\sqrt{\log k}$.
Hint. Apply Borel–Cantelli lemma.

iii) Prove that the series $W(t)$ converges uniformly for $t\in[0,1]$.
Hint. You can follow these steps where $C>0$ is a positive constant.

a) Show that almost surely for n big enough $\max_{2^n \leqslant k < 2^{n+1}} |\xi_k| \leqslant C 2^{\frac{n+1}{4}}$.

b) Show that $\sum_{k=2^n}^{2^{n+1}-1} |s_k(t)| \leqslant 2^{-\frac{n+2}{2}}$.

c) Show that for m big enough $\sum_{k=2^m}^{\infty} |\xi_k||s_k(t)| \leqslant C \sum_{n=m}^{\infty} 2^{-\frac{n+3}{4}}$.

Solution

i) For $0 \leqslant s, t \leqslant 1$, using Exercise 5 we have

$$\begin{aligned}\sum_{k=0}^{\infty} s_k(t) s_k(s) &= \sum_{k=0}^{\infty} \langle \chi_{[0,t]}, h_k \rangle \langle \chi_{[0,s]}, h_k \rangle = \Big\langle \chi_{[0,t]}, \sum_{k=0}^{\infty} \langle \chi_{[0,s]}, h_k \rangle h_k \Big\rangle = \langle \chi_{[0,t]}, \chi_{[0,s]} \rangle \\ &= \min\{s, t\}.\end{aligned}$$

ii) For any $x > 0$ and $k \geqslant 2$ we have

$$P(|\xi_k| > x) = \frac{2}{\sqrt{2\pi}} \int_x^{\infty} e^{-\frac{s^2}{2}} \leqslant \frac{2}{\sqrt{2\pi}} e^{-\frac{x^2}{4}} \int_x^{\infty} e^{-\frac{s^2}{4}} \leqslant \frac{2}{\sqrt{2\pi}} e^{-\frac{x^2}{4}} \int_0^{\infty} e^{-\frac{s^2}{4}}.$$

Hence, setting $x = 4\sqrt{\log k}$ and $C = \frac{2}{\sqrt{2\pi}} \int_0^{\infty} e^{-\frac{s^2}{4}}$, we obtain

$$P(|\xi_k| > 4\sqrt{\log k}) \leqslant C e^{-4 \log k} \leqslant C k^{-4}.$$

Since $\sum k^{-4} < \infty$, Borel–Cantelli lemma implies that $P(|\xi_k| > 4\sqrt{\log k} \text{ i.o.}) = 0$. Consequently, $P(|\xi_k| \leqslant 4\sqrt{\log k} \text{ eventually}) = 1$ and almost surely for any sufficiently large k it holds $|\xi_k| \leqslant 4\sqrt{\log k}$.

iii) First, note that $\sqrt{\log k} \leqslant \sqrt{2} k^{1/4}$ (for all $x, \alpha > 0$ we have $\alpha \log x = \log(x^\alpha) \leqslant x^\alpha$), hence from the previous point we have

$$\max_{2^n \leqslant k < 2^{n+1}} |\xi_k| \leqslant \max_k 4\sqrt{2} k^{1/4} \leqslant C 2^{\frac{n+1}{4}},$$

where $C = 4\sqrt{2}$. Furthermore, note that for any $2^n \leqslant k < 2^{n+1}$ with $n \geqslant 2$ and for all $0 \leqslant t \leqslant 1$, $|s_k(t)| \leqslant 2^{-(n+2)/2}$ (hat functions) and therefore

$$\sum_{k=2^n}^{2^{n+1}-1} |s_k(t)| \leqslant 2^{-\frac{n+2}{2}},$$

as the hat functions have non-overlapping supports. Consequently, for any $0 \leqslant t \leqslant 1$, we bound the remainder by

$$\sum_{k=2^m}^{\infty} |\xi_k||s_k(t)| = \sum_{n=m}^{\infty} \sum_{k=2^n}^{2^{n+1}-1} |\xi_k||s_k(t)| \leqslant C \sum_{n=m}^{\infty} 2^{\frac{n+1}{4}} 2^{-\frac{n+2}{2}} \leqslant C \sum_{n=m}^{\infty} 2^{-\frac{n+3}{4}}.$$

Since the geometric series $\sum (2^{-1/4})^n$ converges, the remainder is smaller than ϵ for sufficiently large m. As this holds for all $0 \leqslant t \leqslant 1$, the series converges uniformly on $[0, 1]$.

Exercise 7.

The family of Schauder functions $\{s_k\}_{k \geqslant 0}$ is defined for $0 \leqslant t \leqslant 1$ as $s_k(t) = \langle \chi_{[0,t]}, h_k \rangle$, where $\{h_k\}_{k \geqslant 0}$ are the Haar functions and $\langle \cdot, \cdot \rangle$ is the inner product in $\mathrm{L}^2(0, 1)$. Then let

$$W(t) = \sum_{k=0}^{\infty} \xi_k s_k(t), \tag{7.1}$$

where $\{\xi_k\}_{k\geqslant 0}$ is a sequence of independent standard Gaussian random variables $\xi_k \sim N(0,1)$.

Show that $W(t)$ is actually a Brownian motion. Equation (7.1) is the Lévy-Ciesielski construction of the Brownian motion.

Solution

To prove $W(\cdot)$ is a Brownian motion, we first note that clearly $W(0)=0$ a.s. We assert as well that $W(t)-W(s)$ is $N(0,t-s)$ for all $0\leqslant s\leqslant t\leqslant 1$. To prove this, let us compute

$$\begin{aligned}
E\Big(e^{i\lambda(W(t)-W(s))}\Big) &= E\Big(e^{i\lambda\sum_{k=0}^{\infty}\xi_k(s_k(t)-s_k(s))}\Big)\\
&= \prod_{k=0}^{\infty} E\Big(e^{i\lambda\xi_k(s_k(t)-s_k(s))}\Big) \quad \text{by independence}\\
&= \prod_{k=0}^{\infty} e^{-\frac{\lambda^2}{2}(s_k(t)-s_k(s))^2} \quad \text{since } \xi_k \text{ is } N(0,1)\\
&= e^{-\frac{\lambda^2}{2}\sum_{k=0}^{\infty}(s_k(t)-s_k(s))^2}\\
&= e^{-\frac{\lambda^2}{2}\sum_{k=0}^{\infty}s_k^2(t)-2s_k(t)s_k(s)+s_k^2(s)}\\
&= e^{-\frac{\lambda^2}{2}(t-2s+s)} \quad \text{by Exercise 6}\\
&= e^{-\frac{\lambda^2}{2}(t-s)}
\end{aligned}\tag{7.2}$$

By uniqueness of characteristic functions, the increment $W(t)-W(s)$ is $N(0,t-s)$, as asserted. Next we claim for all $m=1,2,\ldots$ and for all $0=t_0<t_1<\cdots<t_m\leqslant 1$, that

$$E\Big(e^{i\sum_{j=1}^{m}\lambda_j(W(t_j)-W(t_{j-1}))}\Big) = \prod_{j=1}^{m} e^{-\frac{\lambda_j^2}{2}(t_j-t_{j-1})} \tag{7.3}$$

Once this is proved, we will know from uniqueness of characteristic functions that

$$F_{W(t_1),\ldots,W(t_m)-W(t_{m-1})}(x_1,\ldots,x_m) = F_{W(t_1)}(x_1)\cdots F_{W(t_m)-W(t_{m-1})}(x_m)$$

for all $x_1,\ldots x_m\in\mathbb{R}$. This proves that

$$W(t_1),\ldots,W(t_m)-W(t_{m-1}) \quad \text{are independent.}$$

Thus (7.3) will establish the thesis. Now in the case $m=2$, we have

$$\begin{aligned}
E\Big(e^{i[\lambda_1 W(t_1)+\lambda_2(W(t_2)-W(t_1))]}\Big) &= E\Big(e^{i[(\lambda_1-\lambda_2)W(t_1)+\lambda_2 W(t_2)]}\Big)\\
&= E\Big(e^{i(\lambda_1-\lambda_2)\sum_{k=0}^{\infty}\xi_k s_k(t_1)+i\lambda_2\sum_{k=0}^{\infty}\xi_k s_k(t_2)}\Big)\\
&= \prod_{k=0}^{\infty} E\Big(e^{i\xi_k[(\lambda_1-\lambda_2)s_k(t_1)+\lambda_2 s_k(t_2)]}\Big)\\
&= \prod_{k=0}^{\infty} e^{-\frac{1}{2}((\lambda_1-\lambda_2)s_k(t_1)+\lambda_2 s_k(t_2))^2}\\
&= e^{-\frac{1}{2}\sum_{k=0}^{\infty}(\lambda_1-\lambda_2)^2 s_k^2(t_1)+2(\lambda_1-\lambda_2)\lambda_2 s_k(t_1)s_k(t_2)+\lambda_2^2 s_k^2(t_2)}\\
&= e^{-\frac{1}{2}\left[(\lambda_1-\lambda_2)^2 t_1+2(\lambda_1-\lambda_2)\lambda_2 t_1+\lambda_2^2 t_2\right]} \quad \text{by Ex. 6}\\
&= e^{-\frac{1}{2}[\lambda_1^2 t_1+\lambda_2^2(t_2-t_1)]}
\end{aligned}\tag{7.4}$$

This is (7.3) for $m=2$, and the general case follows similarly.