Series 1 - September 18, 2024

Exercise 1.

Let $(\Omega, \mathcal{F}, P)$ be a probability space
and $\{F_n\}_{n\geqslant 1}$ be a countable sequence of $\mathcal{F}$. Show that:

i) if $F_n \subseteq F_{n+1}$ for all $n \geqslant 1$ then $P\Big(\bigcup_{n=1}^{\infty} F_n\Big) = \lim_{n\to\infty} P(F_n)$,

ii) if $F_n \supseteq F_{n+1}$ for all $n \geqslant 1$ then $P\Big(\bigcap_{n=1}^{\infty} F_n\Big) = \lim_{n\to\infty} P(F_n)$.

Moreover, show that there exists a sequence $\{F'_n\}_{n\geqslant 1}$ such that $F'_i \cap F'_j = \varnothing$ if $i \neq j$ and

$$\bigcup_{n=1}^{\infty} F'_n = \bigcup_{n=1}^{\infty} F_n.$$

Exercise 2.

Let $(\Omega, \mathcal{F}, P)$ be a probability space and $\{A_i\}_{i\geqslant 1}$ be a sequence of events, i.e., $A_i \in \mathcal{F}$.

i) Show that the event $E = \{$“infinitely many A_i occur”$\}$ can be written

$$E = \bigcap_{n=1}^{\infty} \bigcup_{i=n}^{\infty} A_i.$$

Remark. The set E is often called “A_i i.o.”, which means “A_i infinitely often”.

ii) Describe the event $H = \bigcup_{n=1}^{\infty} \bigcap_{i=n}^{\infty} A_i$.

iii) Show that if $\sum_{i=1}^{\infty} P(A_i) < \infty$ then $P(A_i \text{ i.o.}) = 0$ (Borel–Cantelli lemma).

Exercise 3.

Let $(\Omega, \mathcal{F}, P)$ be a probability space and $X\colon \Omega \to \mathbb{R}^n$ be an n-dimensional random variable. Show that $\mathcal{F}(X) = \{X^{-1}(B)\colon B \in \mathcal{B}\}$, where $\mathcal{B}$ is the Borel σ-algebra on $\mathbb{R}^n$, is a σ-algebra. Observe that $\mathcal{F}(X)$ is the smallest σ-algebra with respect to which X is measurable.

Exercise 4.

Let $(\Omega, \mathcal{F}, P)$ be a probability space, $\{X_n\}_{n\geqslant 1}$ be a sequence of real-valued random variables $X_n\colon \Omega \to \mathbb{R}$ and $X\colon \Omega \to \mathbb{R}$ be another real-valued random variable. Denote by F_n and F the distribution function of X_n and X, respectively, and recall the following notions of convergence of random variables:

- $X_n \to X$ in L^2 if $\mathbb{E}[|X_n - X|^2] \to 0$ as $n \to \infty$,
- $X_n \to X$ in probability if for all $\epsilon > 0$ it holds $P(|X_n - X| > \epsilon) \to 0$ as $n \to \infty$,

- $X_n \to X$ in distribution if $\mathbb{E}[g(X_n)] \to \mathbb{E}[g(X)]$ as $n \to \infty$, for any $g \in C_b^0(\mathbb{R})$.

i) Prove that $X_n \to X$ in L^2 implies $X_n \to X$ in probability.

ii) Show that if $X_n \to X$ in probability, then one can extract a subsequence from X_n a.s converging to X. Prove also the following implication: $X_n \to X$ in probability if and only if for every subsequence X_{n_k} one can extract a subsequence from $X_{n_{k_h}}$ a.s converging to X

iii) Prove that $X_n \to X$ in probability implies $X_n \to X$ in distribution.

iv) Show that if $X_n \to X$ in distribution, then $\lim_{n\to\infty} F_n(x) = F(x)$ for all x such that F is continuous *Hint.* Let $\epsilon > 0$ and show that for all $n \geqslant 1$

$$F(x-\epsilon) - P(|X_n - X| > \epsilon) \leqslant F_n(x) \leqslant F(x+\epsilon) + P(|X_n - X| > \epsilon).$$

v) By providing a counterexample show that $X_n \to X$ in probability does not imply $X_n \to X$ in L^2.

vi) By providing a counterexample show that $X_n \to X$ in distribution does not imply $X_n \to X$ in probability.

Exercise 5.

Let $(\Omega, \mathcal{F}, P)$ be a probability space and $\{X_n\}_n$ a sequence of $\mathbb{R}^d$-valued random variables with Gaussian distribution $\mu_{X_n} = \mathcal{N}(m_n, \Sigma_n)$ on $\big(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d)\big)$ for all n.

i) Show that if $m_n \to m$ and $\Sigma_n \to \Sigma$ for $n \to \infty$, then $X_n \to X$ in distribution, where X has distribution $\mathcal{N}(m, \Sigma)$.

ii) Show that if $X_n \to X$ in L^2, then μ_X is Gaussian.

iii) Show that if $X_n \to X$ in distribution, then μ_X is Gaussian.

Hint. Recall that a family of probability measure $\{\mu_n\}_n$ is called tight if for any $\epsilon > 0$, there exists a compact set K_ϵ such that for all μ_n one has $\mu_n(K_\epsilon) > 1 - \epsilon$. Moreover, for $\mathbb{R}^d$-valued random variables, one has that a sequence of weakly convergent probability measure is tight (Prokhorov's Theorem). Use this information to prove by contradiction that both m_n, Σ_n are bounded sequences and contain converging subsequences.

Remark. If $X\colon \Omega \to \mathbb{R}^d$ has Gaussian distribution $\mu_X = \mathcal{N}(m, \Sigma)$, then its characteristic function is

$$\hat{\mu}(t) := \mathbb{E}[e^{iX^\top t}] = \exp\{im^\top t - \frac{1}{2} t^\top \Sigma t\}, \quad \text{for } t \in \mathbb{R}^d. \tag{5.1}$$

Moreover, one has that

- if $\mu_n \rightharpoonup \mu$, (equivalently $X_n \to X$ in distribution), then $\hat{\mu}_n(t) \to \hat{\mu}(t), \quad \forall t \in \mathbb{R}^d$
- if $\hat{\mu}_n(t) \to \psi(t), \quad \forall t \in \mathbb{R}^d$ and $\psi(t)$ is continuous at $t = 0$, then ψ is the characteristic function of a probability distribution μ, and $\mu_n \rightharpoonup \mu$.

Exercise 6.

Let $(\Omega, \mathcal{F}, P)$ be a probability space and $\mathcal{G} \subseteq \mathcal{F}$ be a σ-algebra. Let $X, Y\colon \Omega \to \mathbb{R}$ be integrable random variables on Ω. Using the definition of conditional expectation of random variables given a σ-algebra, prove the following statements.

i) $\mathbb{E}(aX + bY|\mathcal{G}) = a\mathbb{E}(X|\mathcal{G}) + b\mathbb{E}(Y|\mathcal{G})$ a.s. for $a, b \in \mathbb{R}$.

ii) If X is $\mathcal{G}$-measurable then $\mathbb{E}(X|\mathcal{G}) = X$ a.s.

iii) If X is $\mathcal{G}$-measurable and XY is integrable then $\mathbb{E}(XY|\mathcal{G}) = X\mathbb{E}(Y|\mathcal{G})$ a.s.

iv) If X is independent of $\mathcal{G}$ then $\mathbb{E}(X|\mathcal{G}) = \mathbb{E}(X)$ a.s.

v) If $\mathcal{H}$ is a σ-algebra such that $\mathcal{H} \subseteq \mathcal{G}$ then

$$\mathbb{E}(X|\mathcal{H}) = \mathbb{E}\big(\mathbb{E}(X|\mathcal{G})\big|\mathcal{H}\big) = \mathbb{E}\big(\mathbb{E}(X|\mathcal{H})\big|\mathcal{G}\big) \text{ a.s.}$$

vi) If $X \leqslant Y$ a.s. then $\mathbb{E}(X|\mathcal{G}) \leqslant \mathbb{E}(Y|\mathcal{G})$ a.s.