Series 12 - December 11, 2024

Exercise 1.

Consider a numerical scheme of weak order $p \geqslant 1$, delivering a discrete solution $\{X_n\}_{n=0}^N$ on a grid with time step $\Delta t = \frac{T}{N}$, and the Monte Carlo estimator $\hat{Z} = \frac{1}{M}\sum_{i=1}^M \varphi(X_N^{(i)})$ with $X_N^{(i)} \sim X_n$ i.i.d. to approximate $Z = \mathbb{E}[X(T)]$. We define the computational cost of the Monte Carlo estimator $\hat{Z}$ as

$$\text{cost} = \mathcal{O}(\text{number of time steps} \times \text{number of sample paths}) = \mathcal{O}\left(\frac{M}{\Delta t}\right),$$

and we say that the estimator has accuracy $\epsilon > 0$ if $\sqrt{\text{MSE}(\hat{Z})} = \mathcal{O}(\epsilon)$.

i) Compute the optimal values of M and Δt that minimize the computational cost $\eta = M/\Delta t$, subject to a fixed accuracy ϵ; derive the corresponding cost of the Monte Carlo estimator as a function of ϵ in this case. (Hint: Solve a constrained optimization problem in $(M, \Delta t)$, where the variable M is treated as a positive real number)

ii) Implement this estimator to compute $\mathbb{E}[(X(T) - K)_+]$ where X_t solves the SDE

$$dX_t = rX_t dt + \sigma X_t dW_t.$$

Use $T = 1, X_0 = 1, K = 100, r = 0.05, \sigma = 0.01$. Choose a sequence of tollerance $\varepsilon = 0.1, 0.05, 0.025, 0.0125, \ldots$. For each ε, find the "nearly optimal" $M = M(\varepsilon)$ and $\Delta t = \Delta t(\varepsilon)$ and estimate the MSE by repeiting the simulation several times (the exact value of Z can be computed analytically). Plot the (estimated) MSE versus ε^2 and versus η. Comment the results.

Exercise 2.

Implement the MLMC method for the Euler–Maruyama scheme with L levels. Consider the SDE on $[0, T]$

$$\begin{aligned} \mathrm{d}X(t) &= \lambda X(t)\mathrm{d}t + \mu X(t)\mathrm{d}W(t), \\ X(0) &= X_0, \end{aligned} \tag{2.1}$$

with $\lambda = 1$, $\mu = 0.1$, $T = 1$, $X_0 = 0.1$ and the function $\phi(x) = x$. Take a sequence of discretizations with $\Delta t_\ell = 2^{-\ell}$.

i) Plot the variances $\text{Var}(\phi_\ell)$ and $V_\ell := \text{Var}(\phi_\ell - \phi_{\ell-1})$, as well as $B_\ell = |\mathbb{E}[\phi_l - \phi]|$, as a function of the level $\ell = 0, \ldots, 10$. Estimate these quantities by Monte Carlo with sufficient samples.

ii) From the previous point, fit models $V_\ell \approx C_v \Delta t_\ell^\beta$ and $B_\ell \approx C_b \Delta t^\alpha$. Verify that $\beta \approx 1$ and $\alpha \approx 1$. For a fixed $L \in \{3, \ldots, 10\}$ estimate the bias accuracy $\varepsilon = |\mathbb{E}[\phi_L - \phi]|$ and consider the choice of sample sizes $M_\ell = \varepsilon^{-2} L V_\ell \approx L \frac{C_v \Delta_\ell^\beta}{C_b^2 \Delta_\ell^{2\alpha}}$, $\ell = 0, \ldots, L$.

iii) Run several times the MLMC algorithm with the choices of M_ℓ from the previous point. Estimate and plot the MSE of the MLMC method for different values of $L = 3, \ldots, 10$ as a function of the computational cost $\sum_{\ell=0}^L M_\ell \Delta t_\ell^{-1}$. (The exact value $\mathbb{E}[\phi]$ can be computed analytically). Moreover, estimate and plot on the same figure the MSE of the standard Monte–Carlo method corresponding to (roughly) the same computational costs.

iv) For a given L, find optimal values of $\{M_\ell\}_{\ell=0}^L$ that minimize the cost $\sum_{\ell=0}^L M_\ell \Delta t_\ell^{-1}$, subject to a fixed accuracy ε. (Treat the variables M_ℓ as positive real numbers to solve this constrained optimization problem). Derive the correspoding computational cost of the MLMC estimator as a function of ε. Repeat the previous point with this choice of sample sizes and compare the results.

Exercise 3.

Consider the SDE on $[0,1]$

$$\begin{aligned} \mathrm{d}X(t) &= \lambda X(t)\mathrm{d}t + \mu X(t)\mathrm{d}W(t), \\ X(0) &= X_0, \end{aligned} \tag{3.1}$$

where $\lambda, \mu \in \mathbb{R}$ are such that the solution is mean square stable, i.e., $\lambda + \mu^2/2 < 0$ (see Exercise 4 of Series 11). Let $\phi\colon \mathbb{R} \to \mathbb{R}$ be a Lipschitz function and approximate $\phi(X(1))$ with MLMC and the Euler–Maruyama method.

i) The Euler–Maruyama method has a step size restriction when applied to (3.1), i.e., Δt has to be chosen below a threshold Δt_{EM} for the method to be mean-square stable. What is the value of Δt_{EM}? What is the minimum level ℓ_{EM} which can be employed?

The MLMC estimator is then given by

$$\widehat{E} = \sum_{\ell=\ell_{\mathrm{EM}}}^{L} \frac{1}{M_\ell} \sum_{i=1}^{M_\ell} (\phi_\ell^{(i)} - \phi_{\ell-1}^{(i)}).$$

Moreover, we remark that if the most refined level L for attaining the desired tolerance is such that $l_{\mathrm{EM}} \geqslant L$, then a simple Monte Carlo method with time step Δt_{EM} is employed.

ii) Consider the following definition for the number of trajectories

$$M_\ell = \begin{cases} 2^{2L-\ell}(L-\ell_{\mathrm{EM}}) & \text{if } \ell = \ell_{\mathrm{EM}}+1, \dots, L, \\ 2^{2L}(L-\ell_{\mathrm{EM}}) & \text{if } \ell = \ell_{\mathrm{EM}}. \end{cases}$$

How do you choose L such that the MSE of the MLMC estimator is $\mathcal{O}(\varepsilon^2)$? What is the computational cost in this case?

iii) Modify the implementation of MLMC in Exercise 2 to take into account the considerations above. Set $\phi(x) = x$ and apply the method to equation (3.1) with $\lambda \in \{-10, -50, -250\}$, $\mu = \sqrt{-\lambda}$ and $X_0 = 1$. Consider $L \in \{1, 2, \dots, 10\}$ and plot the computational cost as a function of the finest step size Δt_L. What do you observe?

iv) Compare the previous results with those obtained with a standard MLMC applied to the drift implicit Euler Method (stochastic θ-method with $\theta = 1$), using all the levels.