Series 12 - December 11, 2024

Exercise 1.

Consider a numerical scheme of weak order $p \geqslant 1$, delivering a discrete solution $\{X_n\}_{n=0}^N$ on a grid with time step $\Delta t = \frac{T}{N}$, and the Monte Carlo estimator $\hat{Z} = \frac{1}{M}\sum_{i=1}^M \varphi(X_N^{(i)})$ with $X_N^{(i)} \sim X_n$ i.i.d. to approximate $Z = \mathbb{E}[X(T)]$. We define the computational cost of the Monte Carlo estimator $\hat{Z}$ as

$$\text{cost} = \mathcal{O}(\text{number of time steps} \times \text{number of sample paths}) = \mathcal{O}\left(\frac{M}{\Delta t}\right),$$

and we say that the estimator has accuracy $\epsilon > 0$ if $\sqrt{\mathrm{MSE}(\hat{Z})} = \mathcal{O}(\epsilon)$.

i) Compute the optimal values of M and Δt that minimize the computational cost $\eta = M/\Delta t$, subject to a fixed accuracy ϵ; derive the corresponding cost of the Monte Carlo estimator as a function of ϵ in this case. (Hint: Solve a constrained optimization problem in $(M, \Delta t)$, where the variable M is treated as a positive real number)

ii) Implement this estimator to compute $\mathbb{E}[(X(T) - K)_+]$ where X_t solves the SDE

$$dX_t = rX_t dt + \sigma X_t dW_t.$$

Use $T = 1, X_0 = 1, K = 100, r = 0.05, \sigma = 0.01$. Choose a sequence of tollerance $\varepsilon = 0.1, 0.05, 0.025, 0.0125, \ldots$. For each ε, find the "nearly optimal" $M = M(\varepsilon)$ and $\Delta t = \Delta t(\varepsilon)$ and estimate the MSE by repeiting the simulation several times (the exact value of Z can be computed analytically). Plot the (estimated) MSE versus ε^2 and versus η. Comment the results.

Solution

i) If the method has a weak order $p \geqslant 1$, then we have $\mathrm{MSE}(\hat{Z}) = \mathcal{O}(\frac{1}{M} + (\Delta t)^{2p})$. Hence, as we want $\sqrt{\mathrm{MSE}} = \mathcal{O}(\epsilon)$, it is sufficient to have $M^{-1} = \mathcal{O}(\epsilon^2)$ and $\Delta t = \mathcal{O}(\epsilon^{1/p})$. The cost is in this case

$$\text{cost} = \mathcal{O}(\Delta t^{-1} \cdot M) = \mathcal{O}(\epsilon^{-1/p} \cdot \epsilon^{-2}) = \mathcal{O}(\epsilon^{-(2+1/p)}).$$

Assuming that the MSE satisfies $\mathrm{MSE} \leqslant C_1 M^{-1} + C_2(\Delta t)^{2p} \leqslant \varepsilon^2$ for some $C_1, C_2 > 0$, we could define the following optimization problem:

$$\min_{M,\Delta t} \frac{M}{\Delta t} \text{ s.t. } C_1 M^{-1} + C_2(\Delta t)^{2p} \leqslant \varepsilon^2.$$

Defining the Lagrangian $\mathcal{L}(M, \Delta t) = M(\Delta t)^{-1} + \lambda(C_1 M^{-1} + C_2(\Delta t)^{2p})$ we have

$$\frac{\partial \mathcal{L}(M, \Delta t)}{\partial M} = \Delta t^{-1} - \lambda C_1 M^{-2} = 0, \quad \frac{\partial \mathcal{L}(M, \Delta t)}{\partial \Delta t} = M\Delta t^{-2} + 2p\lambda C_2 \Delta^{2p-1} = 0,$$

which implies

$$\lambda = \frac{M^2}{C_1 \Delta t} = \frac{M}{2pC_2(\Delta t)^{2p+1}}$$

and, hence,

$$C_2(\Delta t)^p = \frac{C_1}{M}.$$

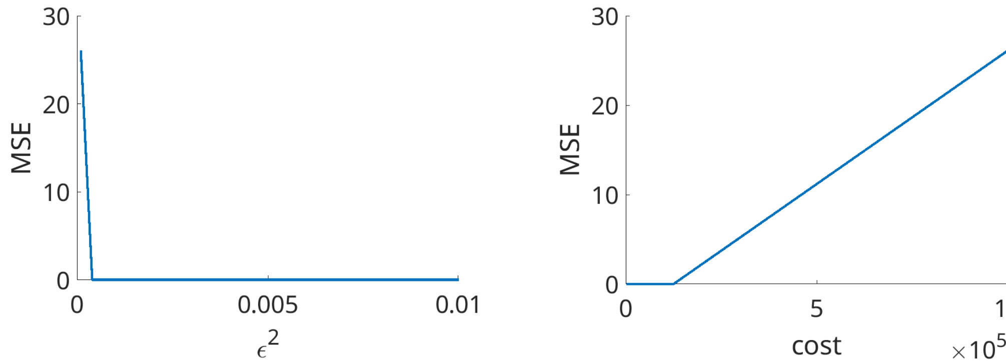

Figure 1: MSE with respect to ε^2 and the cost implementing a Euler-Maruyama method.

This solution is quite obvious, the optimal choice consists in balancing the two terms. Without loss of generalities, assume $C_1 = C_2$. So $\eta = M\Delta t^{-1} = MM^{\frac{1}{2p}} = M^{\frac{1+2p}{2p}}$, which gives $M = \eta^{\frac{2p}{1+2p}}$ and $\Delta t = \eta^{-\frac{1}{1+2p}}$. Therefore, we conclude that $\text{MSE} = \mathcal{O}(M^{-1}) = \mathcal{O}(\eta^{-\frac{2p}{1+2p}})$, which implies $\sqrt{\text{MSE}} = \mathcal{O}(\eta^{-\frac{p}{1+2p}})$.

ii) The plots are shown in Figure

Exercise 2.

Implement the MLMC method for the Euler–Maruyama scheme with L levels. Consider the SDE on $[0, T]$

$$\begin{aligned} \mathrm{d}X(t) &= \lambda X(t)\mathrm{d}t + \mu X(t)\mathrm{d}W(t), \\ X(0) &= X_0, \end{aligned} \tag{2.1}$$

with $\lambda = 1$, $\mu = 0.1$, $T = 1$, $X_0 = 0.1$ and the function $\phi(x) = x$. Take a sequence of discretizations with $\Delta t_\ell = 2^{-\ell}$.

i) Plot the variances $\text{Var}(\phi_\ell)$ and $V_\ell := \text{Var}(\phi_\ell - \phi_{\ell-1})$, as well as $B_\ell = |\mathbb{E}[\phi_l - \phi]|$, as a function of the level $\ell = 0, \dots, 10$. Estimate these quantities by Monte Carlo with sufficient samples.

ii) From the previous point, fit models $V_\ell \approx C_v \Delta t_\ell^\beta$ and $B_\ell \approx C_b \Delta t^\alpha$. Verify that $\beta \approx 1$ and $\alpha \approx 1$. For a fixed $L \in \{3, \dots, 10\}$ estimate the bias accuracy $\varepsilon = |\mathbb{E}[\phi_L - \phi]|$ and consider the choice of sample sizes $M_\ell = \varepsilon^{-2} L V_\ell \approx L \frac{C_v \Delta_\ell^\beta}{C_b^2 \Delta_\ell^{2\alpha}}$, $\ell = 0, \dots, L$.

iii) Run several times the MLMC algorithm with the choices of M_ℓ from the previous point. Estimate and plot the MSE of the MLMC method for different values of $L = 3, \dots, 10$ as a function of the computational cost $\sum_{\ell=0}^{L} M_\ell \Delta t_\ell^{-1}$. (The exact value $\mathbb{E}[\phi]$ can be computed analytically). Moreover, estimate and plot on the same figure the MSE of the standard Monte–Carlo method corresponding to (roughly) the same computational costs.

iv) For a given L, find optimal values of $\{M_\ell\}_{\ell=0}^{L}$ that minimize the cost $\sum_{\ell=0}^{L} M_\ell \Delta t_\ell^{-1}$, subject to a fixed accuracy ε. (Treat the variables M_ℓ as positive real numbers to solve this constrained optimization problem). Derive the correspoding computational cost of the MLMC estimator as a function of ε. Repeat the previous point with this choice of sample sizes and compare the results.

Solution

The requested plots are given in Figure 2. We estimate point *i)* with a Monte-Carlo method of samples $M_l \approx 10^6$ for each l.

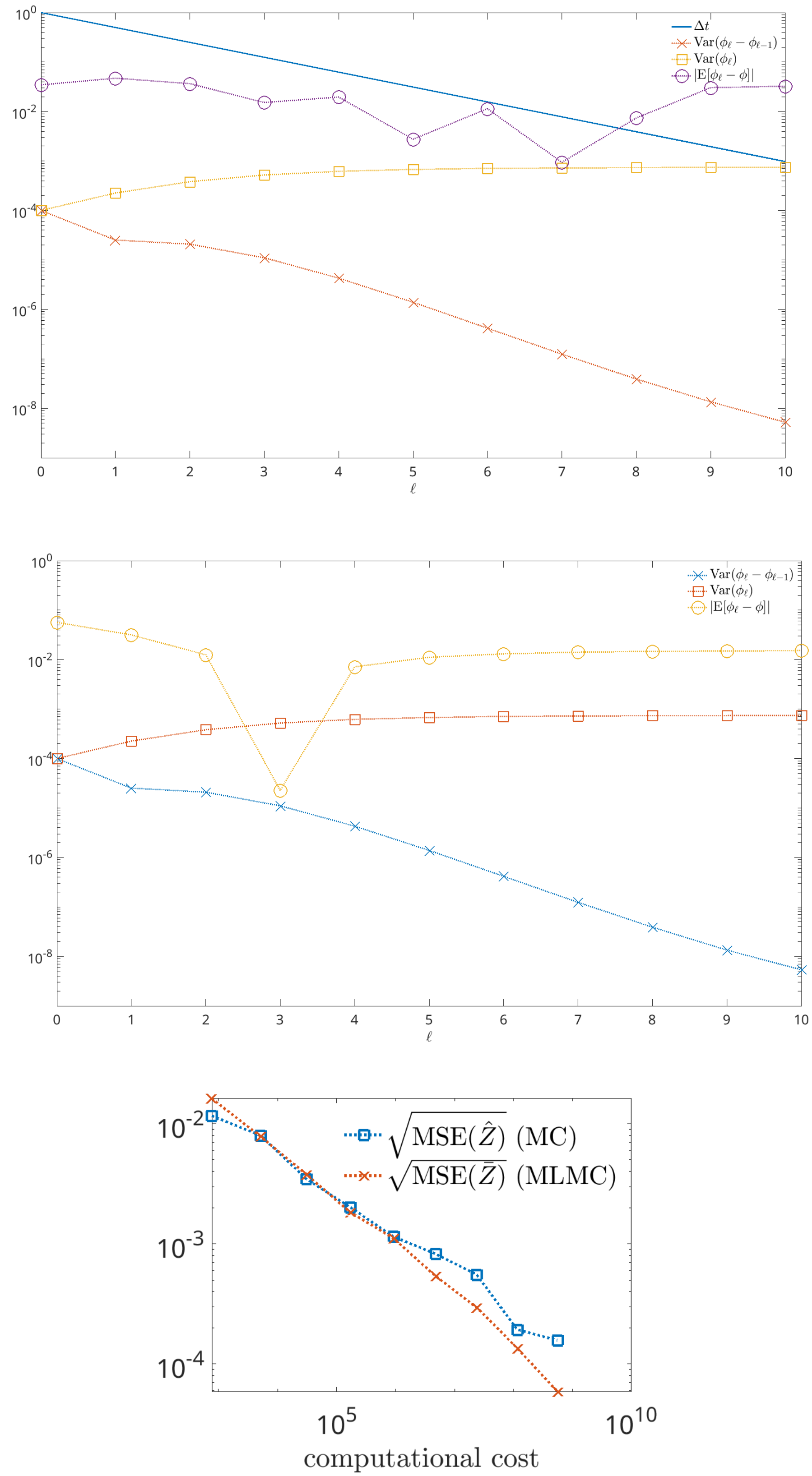

Figure 2: Variances and bias as a function of the level (above - point i), and intermediate - ii)) . MSE as a function of the computational cost (below).

Instead, considering the point *iv*), please refer to the Lecture of the course concerning Monte-Carlo and Multilevel Monte-Carlo, where the expression of the optimal M_l^* is given by

$$M_l^* = \varepsilon^{-2}\sqrt{\frac{v_l}{c_l}}\left(\sum_{k=0}^{L}\sqrt{vkc_k}\right). \tag{2.2}$$

After having substituting the values of c_l and v_l in (2.2), we find that $M_l^* \approx M_\ell$, the chosen sample size of point *ii*).

Exercise 3.

Consider the SDE on $[0, 1]$

$$\begin{aligned} \mathrm{d}X(t) &= \lambda X(t)\mathrm{d}t + \mu X(t)\mathrm{d}W(t), \\ X(0) &= X_0, \end{aligned} \tag{3.1}$$

where $\lambda, \mu \in \mathbb{R}$ are such that the solution is mean square stable, i.e., $\lambda + \mu^2/2 < 0$ (see Exercise 4 of Series 11). Let $\phi\colon \mathbb{R} \to \mathbb{R}$ be a Lipschitz function and approximate $\phi(X(1))$ with MLMC and the Euler–Maruyama method.

i) The Euler–Maruyama method has a step size restriction when applied to (3.1), i.e., Δt has to be chosen below a threshold Δt_{EM} for the method to be mean-square stable. What is the value of Δt_{EM}? What is the minimum level ℓ_{EM} which can be employed?

The MLMC estimator is then given by

$$\widehat{E} = \sum_{\ell=\ell_{\mathrm{EM}}}^{L} \frac{1}{M_\ell}\sum_{i=1}^{M_\ell}(\phi_\ell^{(i)} - \phi_{\ell-1}^{(i)}).$$

Moreover, we remark that if the most refined level L for attaining the desired tolerance is such that $l_{\mathrm{EM}} \geqslant L$, then a simple Monte Carlo method with time step Δt_{EM} is employed.

ii) Consider the following definition for the number of trajectories

$$M_\ell = \begin{cases} 2^{2L-\ell}(L-\ell_{\mathrm{EM}}) & \text{if } \ell = \ell_{\mathrm{EM}}+1, \dots, L, \\ 2^{2L}(L-\ell_{\mathrm{EM}}) & \text{if } \ell = \ell_{\mathrm{EM}}. \end{cases}$$

How do you choose L such that the MSE of the MLMC estimator is $\mathcal{O}(\varepsilon^2)$? What is the computational cost in this case?

iii) Modify the implementation of MLMC in Exercise 2 to take into account the considerations above. Set $\phi(x) = x$ and apply the method to equation (3.1) with $\lambda \in \{-10, -50, -250\}$, $\mu = \sqrt{-\lambda}$ and $X_0 = 1$. Consider $L \in \{1, 2, \dots, 10\}$ and plot the computational cost as a function of the finest step size Δt_L. What do you observe?

iv) Compare the previous results with those obtained with a standard MLMC applied to the drift implicit Euler Method (stochastic θ-method with $\theta = 1$), using all the levels.

Solution

i) The value of Δt_{EM} is given by considering the case $\theta = 0$ in the stochastic θ-method, i.e.

$$\Delta t_{\mathrm{EM}} = -\frac{2\lambda + \mu^2}{\lambda^2},$$

hence, if $\Delta t_{\mathrm{EM}} < 1$, some levels of the standard MLMC procedure are inaccessible and the minimum level is

$$l_{\mathrm{EM}} = \lceil |\log_2(\Delta t_{\mathrm{EM}})| \rceil.$$

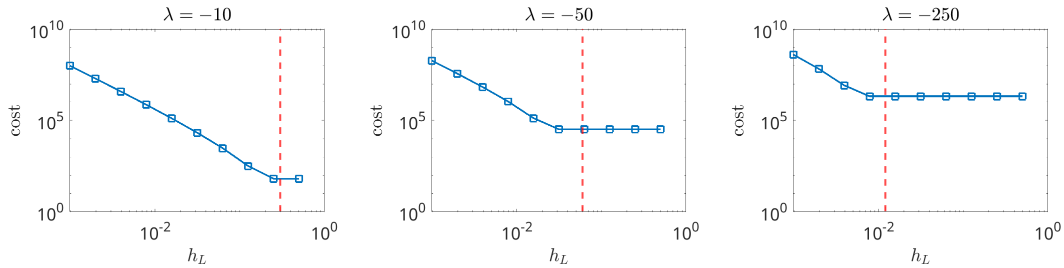

Figure 3: Computational cost as a function of the finest step size for different values of λ in Exercise 3. The red dotted lines represent the values of $\varDelta t_{\mathrm{EM}}$.

ii) By the *bias–variance* decomposition of the MSE, we have

$$\mathrm{MSE}(\hat{E}) = C\varDelta t_L^2 + \mathrm{Var}(\hat{E}),$$

where the variance is given by

$$\mathrm{Var}(\hat{E}) = \frac{\mathrm{Var}(\phi_{l_{\mathrm{EM}}})}{M_{l_{\mathrm{EM}}}} + \sum_{l=l_{\mathrm{EM}}+1}^{L} \frac{\mathrm{Var}(\phi_l - \phi_{l-1})}{M_l}.$$

Plugging the definition of M_l and recalling that $\mathrm{Var}(\phi_l - \phi_{l-1}) \leqslant C\varDelta t_l$, we have

$$\mathrm{Var}(\hat{E}) \leqslant \varDelta t_L^2 \frac{\mathrm{Var}(\phi_{l_{\mathrm{EM}}})}{L - l_{\mathrm{EM}}} + C\varDelta t_L^2.$$

Hence, since $\mathrm{Var}(\phi_{l_{\mathrm{EM}}}) = \mathcal{O}(1)$ in order to have $\mathrm{MSE}(\hat{E}) = \mathcal{O}(\varepsilon^2)$ we can choose $L = |\log_2(\epsilon)|$. The computational cost is then given by

$$\begin{aligned}
\mathrm{Cost}(\hat{E}) &= 2^{2L} 2^{l_{\mathrm{EM}}}(L - l_{\mathrm{EM}}) + \sum_{l=l_{\mathrm{EM}}+1}^{L} 2^l (L - l_{\mathrm{EM}}) 2^{2L-l} \\
&= 2^{2L}(L - l_{\mathrm{EM}})\big((L - l_{\mathrm{EM}}) + 2^{l_{\mathrm{EM}}}\big) \\
&= \varepsilon^{-2} \log_2(\varepsilon)\Big(\frac{l_{\mathrm{EM}}}{L} - 1\Big)\Big(\log_2(\varepsilon)\big(\frac{l_{\mathrm{EM}}}{L} - 1\big) + \varepsilon^{-l_{\mathrm{EM}}/L}\Big) \\
&\leqslant C\varepsilon^{-2}\big(|\log_2(\varepsilon)|^2 + |\log_2(\varepsilon)|\varepsilon^{-l_{\mathrm{EM}}/L}\big).
\end{aligned}$$

We see that for $l_{\mathrm{EM}} \to L$ we have $\mathrm{Cost}(\hat{E}) = \mathcal{O}(\varepsilon^{-3})$ as in the Monte Carlo case. Conversely, for $l_{\mathrm{EM}} \to 0$, the cost approaches the standard MLMC cost.

iii) The plots are given in Figure 3.

iv) The Implicit Euler Method is mean-square stable for each $\varDelta t$ if the original SDE is stable. Therefore, we do not need to modify the MLMC procedure. We choose the M_l as done in Exercise 2 and we consider the same tolerance ϵ of the previous point. The plots are given in Figure 4.

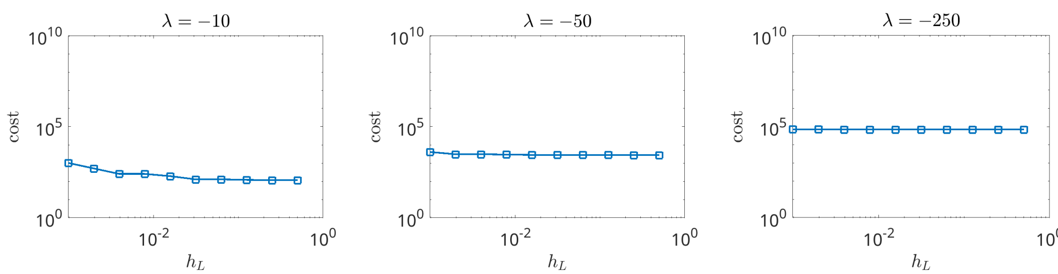

Figure 4: Computational cost for the implementation of MLMC with a Implicit Euler Method as a function of the finest step size for different values of λ in Exercise 3.