Series 11 - December 4, 2024

Exercise 1.

Let $\{X_n\}_{n\geqslant 0}$ be the approximation of SDE

$$\begin{aligned} dX(t) &= \lambda X(t)dt + \mu X(t)dW_t, \quad t \in [0,T], \\ X(0) &= X_0, \end{aligned} \tag{1.1}$$

obtained employing the stochastic θ-method with time step Δt. Consider $T = 500$, $X_0 = 1$, $\lambda = -1.1$, $\mu = 1$. For $\theta = 0, 1/2, 1$, simulate $\mathbb{E}[X_n^2]$ for $\Delta t = 2$ and comment the results.

Solution

For the given data, the stochastic theta method with $\theta = 0$, i.e. the Euler-Maruyama method, is not mean-square stable. In fact, to satisfy this stability definition, the following condition has to be fulfilled

$$\Delta t < \frac{-(2\lambda + \mu^2)}{(1-2\theta)\lambda^2} = 0.99,$$

which is not the case for $\Delta t = 2$.

Exercise 2.

Study the mean-square and asymptotic stability of the simplified weak θ-scheme:

$$Y_{n+1} = Y_n + \theta\mu Y_{n+1}\Delta t + (1-\theta)\mu Y_n \Delta t + \sigma Y_n \Delta \widehat{W}_n \tag{2.1}$$

where $\widehat{W}_n$ is a r.v. such that $\mathbb{P}(\Delta\widehat{W}_n = \pm\sqrt{\Delta t}) = \frac{1}{2}$.

Solution

We can rewrite (2.1) as

$$Y_{n+1}(1-\theta\mu\Delta t) = Y_n\Big(1 + (1-\theta)\mu\Delta t + \sigma\Delta\widehat{W}_n\Big) \tag{2.2}$$

i.e.

$$Y_{n+1} = Y_n\Big(a + b\widehat{V}_n\Big), \tag{2.3}$$

with

$$a := \frac{1+(1-\theta)\mu\Delta t}{(1-\theta\mu\Delta t)}, \quad b := \frac{\sigma\sqrt{\Delta t}}{(1-\theta\mu\Delta t)}, \tag{2.4}$$

and $\mathbb{P}(\widehat{V}_n = \pm 1) = \frac{1}{2}$. The relations (2.4) give us the same conditions as the stochastic theta method for the mean-square stability; in fact

$$\begin{aligned} \mathbb{E}[Y_{n+1}^2] &= \mathbb{E}[Y_n^2](|a|^2 + |b|^2), \\ &= \mathbb{E}[Y_n^2]\left(\frac{|1+(1-\theta)\mu\Delta t|^2 + |\sigma|^2\Delta t}{|1-\theta\mu\Delta t|^2}\right) \end{aligned} \tag{2.5}$$

as $\mathbb{E}[\widehat{V}_n] = 0$ and $\mathbb{E}[\widehat{V}_n^2] = 1$. Therefore, the method (2.1) is mean-square stable if and only if

$$\frac{|1+(1-\theta)\mu\Delta t|^2 + |\sigma|^2\Delta t}{|1-\theta\mu\Delta t|^2} < 1.$$

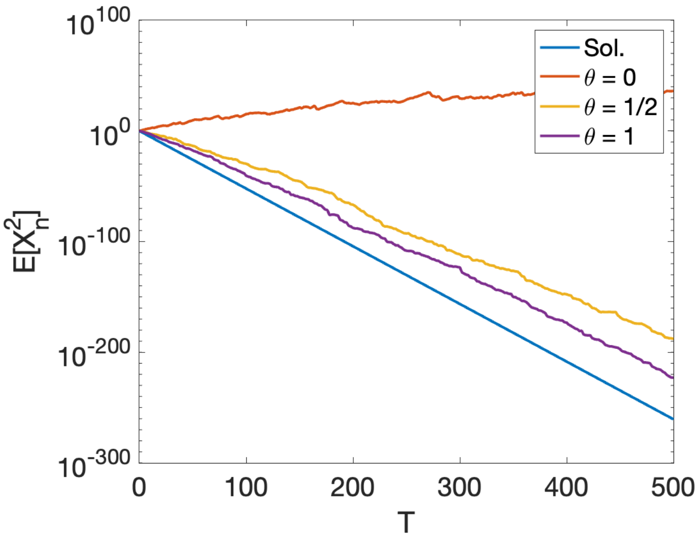

Figure 1: Semilog plot of the mean-Square stability of the stochastic theta method applied to (1.1).

Instead, to consider asymptotically stability for (2.1) we rewrite Y_n as a recurrence

$$Y_n = \left(\prod_{i=0}^{n-1} a + b\hat{V}_i\right) X_0 = \left(\prod_{i=0}^{n-1} V_i\right) X_0 \tag{2.6}$$

assuming $X_0 \geqslant 0$ and $X_0 \neq 0$ with probability 1. Taking logarithms in (2.6) one gets $\log|Y_n| = \log|X_0| + \sum_{i=0}^{n-1}\log|V_i|$. Let $Z_i = \log|V_i|$, the sequence $(Z_i)_i$ satisfies the hypothesis of the law of large numbers, thus

$$\frac{1}{n}\sum_{i=0}^{n-1} Z_i \to \mathbb{E}[Z_1], \quad \text{as } n \to \infty$$

with probability 1. Equation (2.1) is asymptotically stable if and only if $\mathbb{E}[Z_1] < 0$. Therefore

$$\begin{aligned}\mathbb{E}[Z_1] &= \mathbb{E}[\log\left|a + b\hat{V}_1\right|] \\ &= \mathbb{E}[\log\left|a + b\hat{V}_1\right|] = \frac{1}{2}\log|a+b| + \frac{1}{2}\log|a-b| = \frac{1}{2}\log\left|a^2 - b^2\right|\end{aligned} \tag{2.7}$$

Therefore (2.1) is asymptotically stable if and only if $|a^2 - b^2| < 1$.

Exercise 3.

Consider the two-dimensional stochastic differential equation:

$$dX_t = b(X_t)dt + \sigma(X_t)dB_t, \quad X_0 = x_0 \tag{3.1}$$

where B_t is a one-dimensional Brownian motion, and

$$b(x) = \begin{pmatrix} x_2 \cos x_1 \\ 2x_1 \sin x_2 \end{pmatrix}, \quad \sigma(x) = \begin{pmatrix} 3 & -0.3 \\ -0.3 & 3 \end{pmatrix} x.$$

1) Show that b and σ satisfy a local Lipschitz condition as well as a monotonicity condition : $\exists \beta_1, \beta_2 \geqslant 0, \quad x^\top b(x) \leqslant \beta_1 |x|^2,\ |\sigma(x)|^2 \leqslant \beta_2 |x|^2$. Deduce that this has a unique solution in L^p for all $p \geqslant 2$.

2) Show that the solution is exponentially asymptotically stable.

Hint: Consider the Lyapunov function $V(x) = |x|^2 = x_1^2 + x_2^2$ and use the stability condition from Theorem 1 of Lecture 10.

Solution

1) The diffusion term $\sigma(x)$ is globally Lipschitz and satisfy the monotocinity condition as it is a linear function. Concerning $b(x)$, we have

$$\begin{aligned} |b(x) - b(y)| &= \left| \begin{pmatrix} x_2 \cos x_1 \\ 2x_1 \sin x_2 \end{pmatrix} - \begin{pmatrix} y_2 \cos y_1 \\ 2y_1 \sin y_2 \end{pmatrix} \right| = \left| \begin{pmatrix} x_2 \cos x_1 - y_2 \cos y_1 \\ 2x_1 \sin x_2 - 2y_1 \sin y_2 \end{pmatrix} \right| \\ &= \left| \begin{pmatrix} x_2 \cos x_1 - y_2 \cos x_1 + y_2 \cos x_1 - y_2 \cos y_1 \\ 2x_1 \sin x_2 - 2y_1 \sin x_2 + 2y_1 \sin x_2 - 2y_1 \sin y_2 \end{pmatrix} \right| \leqslant \sqrt{2 + 2|y|}|x - y| \end{aligned} \tag{3.2}$$

which implies that b is locally Lipschitz. Moreover

$$x^\top b(x) = \begin{pmatrix} x_1 & x_2 \end{pmatrix} \begin{pmatrix} x_2 \cos x_1 \\ 2x_1 \sin x_2 \end{pmatrix} \quad = x_1 x_2 \cos x_1 + 2x_1 x_2 \sin x_2 \leqslant 4|x_1 x_2| \leqslant 2|x|^2. \tag{3.3}$$

The well-posedness of (3.1) follows from Assumption (C) in Lecture 4. Indeed, one can prove the existence and uniqueness of the solution for (3.1) via a truncating and Borel-Cantelli's argument.

2) Let $V(x,t) = |x|^2$. It is easy to verify that

$$4.29|x|^2 \leqslant LV(x,t) = 2x_1 x_2 \cos x_1 + 4x_1 x_2 \sin x_2 + |\sigma(x)|^2 \leqslant 13.89|x|^2$$

and

$$29.16|x|^2 \leqslant |V_x(x,t)\sigma(x)|^2 = |2x^T \sigma(x)|^2 \leqslant 43.56|x|^4.$$

Applying Theorem 1 we obtain the following lower and upper bound for the sample Lyapunov exponents of the solutions of equation (3.1)

$$-8.745 \leqslant \liminf_{t\to\infty} \frac{1}{t} \log|x(t;t_0,x_0)| \leqslant \limsup_{t\to\infty} \frac{1}{t} \log|x(t;t_0,x_0)| \leqslant -0.345$$

almost surely. Hence the trivial solution of equation (3.1) is almost surely exponentially stable.

Exercise 4.

Consider the linear SDE in $\mathbb{R}^d$:

$$dX_t = FX_t dt + GX_t dB_t, \quad t > 0, \; X_0 = x_0 \tag{4.1}$$

where $F, G \in \mathbb{R}^{d\times d}$ are commuting, diagonalizable matrices (i.e., they are simultaneously diagonalizable, i.e. $\exists V \in \mathbb{R}^{d\times d}$ invertible s.t. $F = VD_F V^{-1}$, $G = VD_G V^{-1}$, with D_F, D_G diagonal).

1) Show that the solution of the solution of (4.1) is given by:

$$X_t = \exp\Big((F - \frac{1}{2}G^2)t + GB_t \Big) x_0$$

Hint. You can verify it directly using Itô's formula. Use the fact that F and G commute.

2) Assume that all the eigenvalues of $F - \frac{1}{2}G^2$ have negative real parts. Show that the solution is exponentially asymptotically stable.

Hint: Make a change of variables $Y_t = V^{-1}X_t$ and show asymptotic stability for Y_t.

3) Find the condition on F and G under which the solution is exponentially mean square stable.

4) Analyze the mean square stability of the stochastic θ-scheme for the SDE (4.1). Use complex numbers for the eigenvalues.

Solution

1) Set

$$Y(t) = \Big(F - \frac{1}{2}G^2\Big)t + GB_t.$$

Define $\varPhi(t) := \exp(Y(t))$. By the commuting condition, we compute the stochastic differential

$$\begin{aligned} d\varPhi(t) &= \exp(Y(t))dY(t) + \frac{1}{2}\exp(Y(t))(dY(t))^2 \\ &= \varPhi(t)dY(t) + \frac{1}{2}\varPhi(t)(G^2)dt \\ &= F\varPhi(t)dt + G\varPhi(t)dB_t. \end{aligned} \tag{4.2}$$

That is, $\varPhi(t)$ satisfies (4.1). Via the existence and uniqueness solution theorem for SDEs, we have that $\varPhi(t)$ is the unique solution of (4.1).

2) As all the eigenvalues of $F - \frac{1}{2}G^2$ have negative real parts, there exists two positive constants C and λ such that

$$\Big|\exp\Big[\Big(F - \frac{1}{2}G^2\Big)t\Big]\Big| \leqslant Ce^{-\lambda t}.$$

Then, it follows that $|Y_t| \leqslant C|x_0|\exp[-\lambda t + \|G\||B_t|]$ and using the property that $\lim\limits_{t\to+\infty} \frac{|B_t|}{t} = 0$ a.s, we have

$$\limsup_{t\to+\infty} \frac{1}{t}\log|Y_t| \leqslant -\lambda.$$

3) Let $Y_t = V^{-1}X_t$. Let μ_i be the eigenvalues of F in the basis V, and σ_i the eigenvalues of G in the same basis, then $Y_t = Dy_0$, $y_0 = V^{-1}x_0$, with $D = \mathrm{diag}(\exp\{\lambda_1\}, \cdots, \exp\{\lambda_d\})$, where $\lambda_i = \Big(\mu_i - \frac{1}{2}\sigma_i^2\Big)t + \sigma_i B_t$ and

$$\|Y_t\|^2 = \sum_i \exp\{2\mathrm{Re}(\lambda_i)\}|Y_{0,i}|^2 = \sum_i \exp\{2\mathrm{Re}\Big(\mu_i - \frac{1}{2}\sigma_i^2\Big)t + \mathrm{Re}(\sigma_i)B_t\}|Y_{0,i}|^2.$$

Therefore,

$$\begin{aligned} \mathbb{E}[\|Y_t\|^2] &= \sum_i \exp\Big(2\mathrm{Re}\Big(\mu_i - \frac{1}{2}\sigma_i^2\Big)t\Big)\mathbb{E}[\exp\{2\mathrm{Re}(\sigma_i)B_t\}][\|y_{0,i}\|^2] \\ &= \sum_i \exp\Big(2\Big(\mathrm{Re}(\mu_i) + \frac{1}{2}\mathrm{Re}(\sigma_i)^2 + \frac{1}{2}\mathrm{Im}(\sigma_i)^2\Big)t\Big)[\|y_{0,i}\|^2] \\ &= \sum_i \exp\Big(2\Big(\mathrm{Re}(\mu_i) + \frac{1}{2}(\sigma_i)^2\Big)t\Big)[\|y_{0,i}\|^2] \end{aligned} \tag{4.3}$$

hence $\mathbb{E}[\|Y_t\|^2] \to 0$ if and only if $\mathrm{Re}(\mu_i) + \frac{1}{2}(\sigma_i)^2 < 0$ for all i.

4) The stochastic θ-method for (4.1) reads as

$$\begin{aligned} Y_{n+1} &= Y_n + \theta FY_{n+1}\varDelta t + (1-\theta)FY_n\varDelta t + GY_n \circ \varDelta W_n \\ &= [I_{d\times d} - \theta F\varDelta t]^{-1}[[I_{d\times d} + (1-\theta)F\varDelta t]Y_n + [I_{d\times d} - \theta F\varDelta t]^{-1}GY_n \circ \varDelta W_n] \\ &= [I_{d\times d} - \theta F\varDelta t]^{-1}[[I_{d\times d} + (1-\theta)F\varDelta t] + [I_{d\times d} - \theta F\varDelta t]^{-1}G \circ \varDelta W_n]Y_n, \end{aligned} \tag{4.4}$$

which is well-defined if $\mathrm{Re}(\mu_i) < 0$ for all i, having

$$[I_{d\times d} - \theta F\varDelta t] = [VV^{-1} - \theta VD_F V^{-1}\varDelta t] = V[I_{d\times d} - \theta D_F\varDelta t]V^{-1}.$$

Considering $V^{-1}Y_n$, squaring and passing to the expectation we arrive to

$$\begin{aligned}
\mathbb{E}\Big[\big|V^{-1}Y_{n+1}\big|^2\Big] &= \mathbb{E}[Y_{n+1}^\top V^{-\top}V^{-1}Y_{n+1}] \\
&= \mathbb{E}[Y_n^\top \Bigg(\underbrace{[I_{d\times d}-\theta F\Delta t]^{-1}[[I_{d\times d}+(1-\theta)F\Delta t]+[I_{d\times d}-\theta F\Delta t]^{-1}G\circ \Delta W_n]}_{=:A}\Bigg)^\top V^{-\top} \\
&\cdot V^{-1}[I_{d\times d}-\theta F\Delta t]^{-1}[[I_{d\times d}+(1-\theta)F\Delta t]+[I_{d\times d}-\theta F\Delta t]^{-1}G\circ \Delta W_n]Y_n] \\
&= \mathbb{E}[\mathbb{E}[Y_n^\top V^{-\top}A^\top V^{-\top}V^{-1}AV^{-1}Y_n]|Y_n] \\
&= \mathbb{E}[Y_n^\top V^{-\top}\mathbb{E}[A^\top V^{-\top}V^{-1}A|Y_n]V^{-1}Y_n]] = \mathbb{E}[Y_n^\top V^{-\top}\mathbb{E}[A^\top V^{-\top}V^{-1}A]V^{-1}Y_n]] \\
&= \mathbb{E}[Y_n^\top V^{-\top}\Big([I_{d\times d}-\theta D_F\Delta t]^{-1}[I_{d\times d}+(1-\theta)D_F\Delta t+D_G\sqrt{\Delta t}]^\top\Big) \\
&\cdot \underbrace{[I_{d\times d}-\theta D_F\Delta t]^{-1}[I_{d\times d}+(1-\theta)D_F\Delta t+D_G\sqrt{\Delta t}]}_{=:B}V^{-1}Y_n] \\
&= \mathbb{E}[Y_n^\top V^{-\top}B^\top BV^{-1}Y_n] = \mathbb{E}\Big[\big|BV^{-1}Y_{n+1}\big|^2\Big]
\end{aligned} \tag{4.5}$$

Therefore, the stochastic θ-method is mean-square stable if and only if

$$|B|_2 = \Big|[I_{d\times d}-\theta D_F\Delta t]^{-1}[I_{d\times d}+(1-\theta)D_F\Delta t+D_G\sqrt{\Delta t}]\Big|_2 < 1. \tag{4.6}$$

Let μ_i be the eigenvalues of F in the basis V, and σ_i the eigenvalues of G in the same basis, then (4.6) is equivalent to

$$\max_i \left|\frac{(1+(1-\theta)\mu_i\Delta t+\sigma_i\sqrt{\Delta t})}{(1-\theta\mu_i\Delta t)}\right| < 1. \tag{4.7}$$